First Steps in Updating Knowing How

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Abstract. We investigate dynamic operations acting over a *knowing how* logic. Our approach makes use of a recently introduced semantics for the *knowing how* operator, based on an indistinguishability relation between plans. This semantics is arguably closer to the standard presentation of *knowing that* modalities in classic epistemic logic. Here, we discuss how the semantics enables us to define dynamic modalities representing different ways in which an agent can learn how to achieve a goal. In this regard, we study two types of updates: ontic updates (for which we provide axiomatizations over a particular class of models), and epistemic updates (for which we investigate some semantic properties).

1 Introduction

Over the last years, a new family of epistemic languages for reasoning about *knowing how* assertions [8] have received much attention. Intuitively, an agent knows how to achieve φ given ψ if she has at her disposal a suitable *course of action* guaranteeing that φ will be the case, whenever she is in a situation in which ψ holds. The concept of knowing how is important not only from a philosophical perspective, but also from a computer science point of view. For instance, it can be seen as a formal account for automated planning and strategic reasoning in AI (see, e.g., [2]).

Most traditional approaches for representing *knowing how* rely in connecting logics of *knowing that* with logics of action (see, e.g., [22,18,14]). However, while a combination of operators for *knowing that* and *ability* (e.g., [26]) produces a *de dicto* concept (*"the agent knows she has an action that guarantees the goal"*), a proper notion of *"knowing how to achieve* φ*"* requires a *de re* clause (*"the agent has an action that she knows guarantees the goal"*; see [15,13] for a discussion). Based on these considerations, [31,32] introduced a new framework based on a *knowing how* binary modality $\mathsf{Kh}(\psi, \varphi)$. At the semantic level, this language is interpreted over relational models — called in this context labeled transition systems (LTSs). In these models, relations describe the actions an agent has at her disposal (in some sense, her *abilities*). Then, $\mathsf{Kh}(\psi, \varphi)$ holds if and only if there is a "proper plan" (a sequence of actions satisfying certain contraints) in the LTS that unerringly leads from every ψ-state only to φ-states.

While variants of this idea have been explored in the literature (see, for instance, [19,20,9,30]), most of them share a fundamental characteristic: relations are interpreted as the agent's available actions; and the abilities of an agent depend *only* on what these actions can achieve. The framework presented in [5] changed this underlying idea by adding a notion of 'indistinguishability' between plans, related to the notion of *strategy indistinguishability* of, e.g., [16,7]. The intuitive idea is, first, that some plans might not be available to the agent. More importantly, she might consider some of them *indistinguishable* from some others. In such cases, having a proper plan σ that leads from any ψ-state to only φ-states is not enough. Instead, the agent also needs for *all her available plans that she cannot distinguish from* σ to satisfy such requirements. As argued in [5], the benefits of these new semantics are threefold. First, it provides an epistemic 'indistinguishability-based' view of an agent's abilities. Second, it enables us to deal with multi-agent scenarios in a more natural way. Third, this new perspective leads to a natural definition of operators that represent dynamic aspects of *knowing how*, more aligned with *dynamic epistemic logic* (DEL; [28]).

This paper focuses on the latter point. We will make use of the indistinguishability-based semantics to investigate some dynamic operators describing changes in the agents' abilities, and hence in their corresponding epistemic states. To the best of our knowledge, this is the first time in which this problem is addressed (except by the brief discussion introduced in [32] about announcements in the context of knowing how). We start by investigating operators that restrict the models based on some sort of *announcement*, in the spirit of [24]. However, as we will see, this kind of updates in the context of knowing how can be seen as ontic updates, rather than epistemic. Then, we will exploit the provided semantics in order to define operations that perform actual epistemic updates. In particular, we will discuss how the indistinguishability relation between plans can be refined in order to perform an epistemic change. We consider our work as the first step towards a dynamic epistemic theory over *knowing how* logics.

Outline. The paper is organized as follows. Sec. 2 recalls the syntax, semantics and a complete axiomatization of the multi-agent *knowing how* logic from [5], discussing also a corresponding notion of bisimulation. These notions are useful in the rest of the paper. Then, Sec. 3 is devoted to investigate different dynamic operators for updating knowing how. First, we introduce ontic updates, based on public announcements [24] and arrow updates [17]. We discuss the properties of the operations, and provide reduction axioms. Then, we provide alternatives for epistemic updates, and discuss some of their semantic properties. In Sec. 4 we offer some final remarks and discuss future lines of work.

2 Basic Definitions

Throughout the text, let Prop be a countable set of propositional symbols, Act a denumerable set of action symbols, and Agt a non-empty finite set of agents.

Definition 1. *Formulas of the language* $\mathsf{L}_{\mathsf{Kh}_i}$ *are given by*

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \mathsf{Kh}_i(\varphi, \varphi),$$

with $p \in \mathsf{Prop}$ and $i \in \mathsf{Agt}$. Other Boolean connectives are defined as usual. Formulas of the form $\mathsf{Kh}_i(\psi, \varphi)$ are read as "when ψ is the case, the agent i knows how to make φ true"*. Define also $\mathsf{A}\varphi := \bigvee_{i \in \mathsf{Agt}} \mathsf{Kh}_i(\neg\varphi, \bot)$ and $\mathsf{E}\varphi := \neg\mathsf{A}\neg\varphi$; they will turn out to be the global universal and existential modalities, resp.*

In [31,32], formulas are interpreted over *labeled transition systems* (LTSs): relational models in which each (basic) relation indicates the source and target of a particular type of action the agent can perform. In the setting introduced in [5], LTSs are extended with a notion of *uncertainty* between plans.

Definition 2 (Actions and plans). *Let Act^* be the set of finite sequences over Act. Elements of Act^* are called* plans*, with ϵ being the* empty plan*. Given $\sigma \in \mathsf{Act}^*$, let $|\sigma|$ be the length of σ (note: $|\epsilon| := 0$). For $0 \leq k \leq |\sigma|$, the plan σ_k is σ's initial segment up to (and including) the kth position (with $\sigma_0 := \epsilon$). For $0 < k \leq |\sigma|$, the action $\sigma[k]$ is the one in σ's kth position.*

Definition 3 (Uncertainty-based LTS). *An* uncertainty-based LTS ($\mathrm{LTS}^{\mathrm{U}}$) *for Prop, Act and Agt is a tuple $\mathcal{M} = \langle \mathrm{W}, \mathrm{R}, \mathbb{S}, \mathrm{V} \rangle$ where: W is a non-empty set of states (called the domain, and denoted by $\mathrm{D}_{\mathcal{M}}$); $\mathrm{R} = \{\mathrm{R}_a \subseteq \mathrm{W} \times \mathrm{W} \mid a \in \mathsf{Act}\}$ is a collection of binary relations on W; $\mathbb{S} = \{\mathbb{S}_i \subseteq 2^{\mathsf{Act}^*} \setminus \{\emptyset\} \mid i \in \mathsf{Agt}\}$ assigns to every agent a non-empty collection of pairwise disjoint non-empty sets of plans: **(i)** $\mathbb{S}_i \neq \emptyset$, **(ii)** $\pi_1, \pi_2 \in \mathbb{S}_i$ with $\pi_1 \neq \pi_2$ implies $\pi_1 \cap \pi_2 = \emptyset$, and **(iii)** $\emptyset \notin \mathbb{S}_i$; and $\mathrm{V} : \mathrm{W} \to 2^{\mathsf{Prop}}$ is a labeling function. Given an $\mathrm{LTS}^{\mathrm{U}}$ $\mathcal{M}$ and $w \in \mathrm{D}_{\mathcal{M}}$, the pair $(\mathcal{M}, w)$ (parenthesis usually dropped) is called a* pointed $\mathrm{LTS}^{\mathrm{U}}$.

Intuitively, $\mathrm{P}_i = \bigcup_{\pi \in \mathbb{S}_i} \pi$ is the set of plans that agent i has at her disposal, and each $\pi \in \mathbb{S}_i$ is an indistinguishability class. Note that, as discussed in [5], there is a one-to-one correspondence between each $\mathbb{S}_i$ and an 'indistinguishability relation' $\sim_i \subseteq \mathrm{P}_i \times \mathrm{P}_i$ describing the agent's *uncertainty* over her available plans ($\sigma_1 \sim_i \sigma_2$ iff there is $\pi \in \mathbb{S}_i$ such that $\{\sigma_1, \sigma_2\} \subseteq \pi$). The presentation used here simplifies the definitions that will follow.

Given her uncertainty over Act^*, the abilities of an agent i depend not on what a single plan can achieve, but rather on what a set of them can guarantee.

Definition 4. *Given $\mathrm{R} = \{\mathrm{R}_a \subseteq \mathrm{W} \times \mathrm{W} \mid a \in \mathsf{Act}\}$ and $\sigma \in \mathsf{Act}^*$, define $\mathrm{R}_\sigma \subseteq \mathrm{W} \times \mathrm{W}$ in the standard way. Then, for $\pi \subseteq \mathsf{Act}^*$ and $U \cup \{u\} \subseteq \mathrm{W}$, define $\mathrm{R}_\pi := \bigcup_{\sigma \in \pi} \mathrm{R}_\sigma$, $\mathrm{R}_\pi(u) := \bigcup_{\sigma \in \pi} \mathrm{R}_\sigma(u)$, and $\mathrm{R}_\pi(U) := \bigcup_{u \in U} \mathrm{R}_\pi(u)$.*

Definition 5 (Strong executability of plans). *Let $\mathcal{M} = \langle \mathrm{W}, \mathrm{R}, \mathbb{S}, \mathrm{V} \rangle$ be an $\mathrm{LTS}^{\mathrm{U}}$, with $\mathrm{R} = \{\mathrm{R}_a \subseteq \mathrm{W} \times \mathrm{W} \mid a \in \mathsf{Act}\}$. A* plan *$\sigma \in \mathsf{Act}^*$ is* strongly executable *(SE) at $u \in \mathrm{W}$ if and only if $v \in \mathrm{R}_{\sigma_k}(u)$ implies $\mathrm{R}_{\sigma[k+1]}(v) \neq \emptyset$ for every $k \in [0 \,..\, |\sigma| - 1]$. We define the set $\mathrm{SE}^{\mathcal{M}}(\sigma) := \{w \in \mathrm{W} \mid \sigma$ is SE at $w\}$. Then, a* set of plans *$\pi \subseteq \mathsf{Act}^*$ is* strongly executable *at $u \in \mathrm{W}$ if and only if* every *plan $\sigma \in \pi$ is* strongly executable *at u. Hence, $\mathrm{SE}^{\mathcal{M}}(\pi) = \bigcap_{\sigma \in \pi} \mathrm{SE}(\sigma)$ is the set of the states in W where π is strongly executable.*

Thus, a plan is strongly executable (at a state) when *all* its partial executions can be completed. Then, a set of plans is strongly executable when *all* its plans are strongly executable. When the model is clear from the context, we will drop the superscript $\mathcal{M}$ and write simply $\mathrm{SE}(\sigma)$ and $\mathrm{SE}(\pi)$.

Now, we have all the ingredients to define the semantics of the logic.

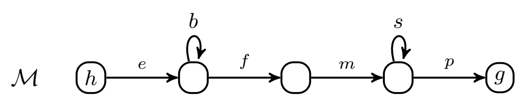
Definition 6. *Let $\mathcal{M} = \langle \mathrm{W}, \mathrm{R}, \{\mathbb{S}_i\}_{i \in \mathsf{Agt}}, \mathrm{V} \rangle$ be an $\mathrm{LTS}^{\mathrm{U}}$; take $w \in \mathrm{W}$. The satisfiability relation $\models$ for $\mathsf{L}_{\mathsf{Kh}_i}$ is inductively defined as:*

$$\begin{array}{lll}
\mathcal{M}, w \models p & \textit{iff}_{\textit{def}} & p \in \mathrm{V}(w) \\
\mathcal{M}, w \models \neg\varphi & \textit{iff}_{\textit{def}} & \mathcal{M}, w \not\models \varphi \\
\mathcal{M}, w \models \psi \vee \varphi & \textit{iff}_{\textit{def}} & \mathcal{M}, w \models \psi \textit{ or } \mathcal{M}, w \models \varphi \\
\mathcal{M}, w \models \mathsf{Kh}_i(\psi, \varphi) & \textit{iff}_{\textit{def}} & \textit{there is } \pi \in \mathbb{S}_i \textit{ s.t.} \\
 & & \quad \textbf{\textit{(i)}}\ [\![\psi]\!]^{\mathcal{M}} \subseteq \mathrm{SE}(\pi) \\
 & & \quad \textbf{\textit{(ii)}}\ \mathrm{R}_\pi([\![\psi]\!]^{\mathcal{M}}) \subseteq [\![\varphi]\!]^{\mathcal{M}},
\end{array}$$

where: $[\![\chi]\!]^{\mathcal{M}} := \{w \in \mathrm{W} \mid \mathcal{M}, w \models \chi\}$. Define: $\mathcal{M} \models \varphi$ iff $[\![\varphi]\!]^{\mathcal{M}} = \mathrm{W}$, and $\models \varphi$ iff $\mathcal{M} \models \varphi$, for all $\mathrm{LTS}^{\mathrm{U}}$ $\mathcal{M}$.

Note: the above-defined modalities A and E are indeed the global modalities from [11]. Indeed, for every model $\mathcal{M}$ and every state w, $\mathcal{M}, w \models \mathsf{A}\varphi$ holds if and only if φ is true in all states in $\mathcal{M}$ [5].

Example 1. Let us consider a simplified scenario for baking a cake, with two agents i and j. The two agents attempt to produce a good cake (represented by the propositional symbol g). Suppose that they are following a similar recipe, and that they have all the ingredientes (h). The recipe states that g is achieved via the following steps: adding eggs (e), beating the eggs (b), adding flour (f), adding milk (m), stir (s) and finally, bake the preparation (p). Thus, the plan needed to achieve g is *ebfmsp*. Agent i, who is an experienced chef, is aware that is the way to get a good cake. On the other hand, agent j has no cooking experience, so she considers that the order in the instructions do not matter.



$\mathbb{S}_i = \{\,\{ebfmsp\}\,\}$

$\mathbb{S}_j = \{\,\{ebfmsp, ebmfsp\}\,\}$

The diagram shows, on the right, the set of indistinguishable plans in $\mathbb{S}_i$ and in $\mathbb{S}_j$. Notice that agent i knows how to get a good cake, provided that she has all the ingredients (i.e., $\mathcal{M} \models \mathsf{Kh}_i(h, g)$). This is due to the fact that agent i *distinguishes ebfmsp* as the "good plan". On the other hand, as j considers that adding milk and adding flour can be done in any order, we have $\mathcal{M} \not\models \mathsf{Kh}_j(h, g)$.

Bisimulations. Bisimulation is a crucial tool for understanding a formal language's expressive power. Here we introduce a generalization of the ideas from [10], now for $\mathsf{L}_{\mathsf{Kh}_i}$ over $\mathrm{LTS}^{\mathrm{U}}$s.

Definition 7. *Let $\mathcal{M} = \langle \mathrm{W}, \mathrm{R}, \{\mathbb{S}_i\}_{i \in \mathsf{Agt}}, \mathrm{V} \rangle$ be an $\mathrm{LTS}^{\mathrm{U}}$ over Prop, Act and Agt. Take $\pi \in 2^{(\mathsf{Act}^*)}$, $U, T \subseteq \mathrm{W}$ and $i \in \mathsf{Agt}$.*

- *Write $U \overset{\pi}{\Rightarrow} T$ iff$_{\textit{def}}$ $U \subseteq \mathrm{SE}(\pi)$ and $\mathrm{R}_\pi(U) \subseteq T$.*
- *Write $U \overset{i}{\Rightarrow} T$ iff$_{\textit{def}}$ there is $\pi \in \mathbb{S}_i$ such that $U \overset{\pi}{\Rightarrow} T$.*

Axioms	Taut	$\vdash \varphi$ for φ a propositional tautology
	DistA	$\vdash \mathsf{A}(\varphi \to \psi) \to (\mathsf{A}\varphi \to \mathsf{A}\psi)$
	TA	$\vdash \mathsf{A}\varphi \to \varphi$
	4KhA	$\vdash \mathsf{Kh}_i(\psi, \varphi) \to \mathsf{A}\mathsf{Kh}_i(\psi, \varphi)$
	5KhA	$\vdash \neg\mathsf{Kh}_i(\psi, \varphi) \to \mathsf{A}\neg\mathsf{Kh}_i(\psi, \varphi)$
	KhE	$\vdash (\mathsf{E}\psi \land \mathsf{Kh}_i(\psi, \varphi)) \to \mathsf{E}\varphi$
	KhA	$\vdash (\mathsf{A}(\chi \to \psi) \land \mathsf{Kh}_i(\psi, \varphi) \land \mathsf{A}(\varphi \to \theta)) \to \mathsf{Kh}_i(\chi, \theta)$
Rules	MP	From $\vdash \varphi$ and $\vdash \varphi \to \psi$ infer $\vdash \psi$
	NecA	From $\vdash \varphi$ infer $\vdash \mathsf{A}\varphi$

Table 1: Axiomatization $\mathcal{L}_{\mathsf{Kh}_i}$ for $\mathsf{L}_{\mathsf{Kh}_i}$ w.r.t. $\mathrm{LTS}^{\mathrm{U}}s$.

Additionally, $U \subseteq \mathrm{W}$ is propositionally definable in $\mathcal{M}$ if and only if there is a propositional formula φ such that $U = [\![\varphi]\!]^{\mathcal{M}}$.

Definition 8 ($\mathsf{L}_{\mathsf{Kh}_i}$-bisimulation). *Let $\mathcal{M} = \langle \mathrm{W}, \mathrm{R}, \{\mathbb{S}_i\}_{i \in \mathsf{Agt}}, \mathrm{V}\rangle$ and $\mathcal{M}' = \langle \mathrm{W}', \mathrm{R}', \{\mathbb{S}'_i\}_{i \in \mathsf{Agt}}, \mathrm{V}'\rangle$ be $\mathrm{LTS}^{\mathrm{U}}s$. A non-empty $Z \subseteq \mathrm{W} \times \mathrm{W}'$ is called an $\mathsf{L}_{\mathsf{Kh}_i}$-bisimulation between $\mathcal{M}$ and $\mathcal{M}'$ if and only if wZw' implies all of the following.*

- ***Atom**: $\mathrm{V}(w) = \mathrm{V}'(w')$.*
- *Kh_i-**Zig**: for any* propositionally *definable $U \subseteq \mathrm{W}$, if $U \overset{i}{\Rightarrow} T$ for some $T \subseteq \mathrm{W}$, then there is $T' \subseteq \mathrm{W}'$ s.t. 1) $Z(U) \overset{i}{\Rightarrow} T'$, and 2) $T' \subseteq Z(T)$.*
- *Kh_i-**Zag**: analogous to Kh_i-**Zig**.*
- *A-**Zig**: for all $u \in \mathrm{W}$ there is a $u' \in \mathrm{W}'$ such that uZu'.*
- *A-**Zag**: for all $u' \in \mathrm{W}'$ there is a $u \in \mathrm{W}$ such that uZu'.*

We write $\mathcal{M}, w \leftrightarrows \mathcal{M}', w'$ when there is an $\mathsf{L}_{\mathsf{Kh}_i}$-bisimulation Z between $\mathcal{M}$ and $\mathcal{M}'$ such that wZw'.

Theorem 1. *Let $\mathcal{M}, w$ and $\mathcal{M}', w'$ be two $\mathrm{LTS}^{\mathrm{U}}s$. $\mathcal{M}, w \leftrightarrows \mathcal{M}', w'$ implies $\mathcal{M}, w \models \varphi$ iff $\mathcal{M}', w' \models \varphi$, for all $\mathsf{L}_{\mathsf{Kh}_i}$-formula φ.*

Axiomatization. We finish this section by recalling an axiom system for $\mathsf{L}_{\mathsf{Kh}_i}$.

Theorem 2 ([5]). *The axiom system from Table 1 is sound and strongly complete w.r.t. the class of all $\mathrm{LTS}^{\mathrm{U}}s$.*

3 Dynamic Knowing How Logics

In this section we will explore different ways in which a *dynamic* operation can be added to $\mathsf{L}_{\mathsf{Kh}_i}$. We can consider a dynamic operator as the indication of performing an update on a model, so that the evaluation of the formula should continue in the modifed model. Some of these model transformations can be interpreted as actions that affect the agents' abilities or her epistemic state. In this section we explore some of these alternatives.

There are at least two ways in which an agent's information might change. It might change because the world changes and she observes this (the *belief update* of the belief change literature; [12]), and it might change because she receives information about the world while the world remains the same (the *belief revision* of the belief change literature; [12]). The former can be called *ontic* change, whereas the latter can be called *epistemic* change. Within dynamic epistemic logic, the first can be represented by a change in valuation, while the second can be represented by changes in the agents' uncertainty [27].

In an $\mathrm{LTS}^{\mathrm{U}}$ $\mathcal{M} = \langle \mathrm{W}, \mathrm{R}, \{\mathbb{S}_i\}_{i \in \mathsf{Agt}}, \mathrm{V}\rangle$, there is a clear distinction between ontic and epistemic information. On the one hand, while R provides *ontic*, *objective* information indicating what the actions themselves can achieve, V describes the actual propositions being true at each state. On the other hand, the *epistemic* state of an agent i (w.r.t. her *knowing how* capabilities) is given by her indistinguishability relation over plans (the set $\mathbb{S}_i$ at her disposal). Hence, in what follows we will consider both ontic and epistemic updates.

3.1 Ontic Updates via Public Announcements

Consider first a model operation removing states (and thus updating the relations). Within the DEL literature, this is interpreted as a *public announcement* (PAL; [24]): an *epistemic* action through which agents get to know publicly that the announced formula is true. Such an *update* model operation is described with the operator $[\chi]$, semantically interpreted as

$$\mathcal{M}, w \models [\chi]\varphi \quad \textit{iff} \quad \mathcal{M}, w \models \chi \text{ implies } \mathcal{M}_\chi, w \models \varphi,$$

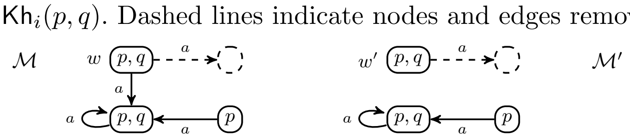
with $\mathcal{M}_\chi$ being the submodel of $\mathcal{M}$ that arises from taking $[\![\chi]\!]^{\mathcal{M}}$ as the new domain, and with the relations and the valuation restricted accordingly.

In the original *knowing how* setting from [31], the relations define the agent's abilities. Thus, an update corresponds to both an ontic and an epistemic change (available actions change, and hence so do the agent's abilities). However, in the $\mathrm{LTS}^{\mathrm{U}}$-based semantics, relations provide only ontic information; thus, an update operation produces an *ontic* change, but not an epistemic one.

The update operator adds expressivity to our $\mathsf{L}_{\mathsf{Kh}_i}$ (a similar result was established in [32] for a Kh modality with intermediate constraints).

Proposition 1. *Adding* $[\chi]$ *to* $\mathsf{L}_{\mathsf{Kh}_i}$ *increases its expressive power.*

Proof. The two $\mathrm{LTS}^{\mathrm{U}}$s $\mathcal{M}$ and $\mathcal{M}'$ (with $\mathbb{S}_i = \mathbb{S}'_i = \{\{a\}\}$) below are bisimilar and hence indistinguishable in $\mathsf{L}_{\mathsf{Kh}_i}$. However, $\mathcal{M}, w \models [p]\mathsf{Kh}_i(p,q)$ whereas $\mathcal{M}', w' \not\models [p]\mathsf{Kh}_i(p,q)$. Dashed lines indicate nodes and edges removed after $[p]$.



A consequence of Prop. 1 is that the modality for PAL-like updates is not reducible to the base logic. This makes sense, as the underlying static logic ($\mathsf{L}_{\mathsf{Kh}_i}$) only expresses properties relative to the existence of a way to achieve certain target states from certain origin states. There is no way to characterize the updates produced by $[\chi]$ with the expressive power provided by the Kh_i modality. This is in contrast with what happens when these modalities are added to standard epistemic logic, where reduction axioms can be defined (see, e.g., [28]).

Is it possible to define an alternative, PAL-like update operator, for which reduction axioms exists in $\mathsf{L}_{\mathsf{Kh}_i}$? We will answer this question below.

Definition 9. *Formulas of the language* $\mathsf{PAL}_{\mathsf{Kh}_i}$ *are given by*

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \mathsf{Kh}_i(\varphi, \varphi) \mid [!\varphi]\varphi,$$

with $p \in \mathsf{Prop}$ *and* $i \in \mathsf{Agt}$.

Definition 10. *Let* $\mathcal{M} = \langle \mathrm{W}, \mathrm{R}, \mathbb{S}, \mathrm{V}\rangle$ *be an* $\mathrm{LTS}^{\mathrm{U}}$*, and let* χ *be a* $\mathsf{PAL}_{\mathsf{Kh}_i}$*-formula. We define* $\mathcal{M}_{!\chi} = \langle \mathrm{W}_{!\chi}, \mathrm{R}_{!\chi}, \mathbb{S}_{!\chi}, \mathrm{V}_{!\chi}\rangle$*, where:*

- $\mathrm{W}_{!\chi} = [\![\chi]\!]^{\mathcal{M}}$,
- $(\mathrm{R}_{!\chi})_a = \{(w, v) \in \mathrm{R}_a \mid w \in [\![\chi]\!]^{\mathcal{M}},\ \mathrm{R}_a(w) \subseteq [\![\chi]\!]^{\mathcal{M}}\}$ *for every* $a \in \mathsf{Act}$,
- $\mathbb{S}_{!\chi} = \mathbb{S}$, *and* $\mathrm{V}_{!\chi}(w) = \mathrm{V}(w)$.

We extend the satisfaction relation $\models$ *from Def. 6 with the case:*

$$\mathcal{M}, w \models [!\chi]\varphi \quad \textit{iff} \quad \mathcal{M}, w \models \chi \textit{ implies } \mathcal{M}_{!\chi}, w \models \varphi.$$

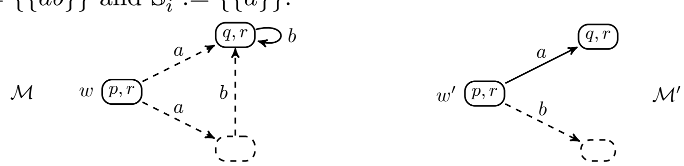
The only difference between the $\mathcal{M}_{!\chi}$ introduced above and the standard $\mathcal{M}_\chi$ (which is the restriction of $\mathcal{M}$ to the states satisfying χ) is in the definition of the relations. In the proposal here, a stronger condition is needed for an a-edge from a state $w \in [\![\chi]\!]^{\mathcal{M}}$ to survive after the update: if $\mathrm{R}_a(w) \not\subseteq [\![\chi]\!]^{\mathcal{M}}$ then $(\mathrm{R}_{!\chi})_a(w) = \emptyset$, but if $\mathrm{R}_a(w) \subseteq [\![\chi]\!]^{\mathcal{M}}$ then $(\mathrm{R}_{!\chi})_a(w) = \mathrm{R}_a(w)$. Notice that in this context, the elimination of some states indicates that the situations they describe are no longer *reachable*, rather than no longer *possible*.

The two forms of model update discussed above bear a resemblance to the two forms of updating neighbourhood models from [21]. Recall that a neighbourhood model [25,23] is given by: a non-empty domain W, an atomic valuation, and a neighbourhood function $\mathrm{N} : \mathrm{W} \to 2^{2^{\mathrm{W}}}$, assigning a set of sets of states to each possible state. Let $U \subseteq \mathrm{W}$ be a non-empty set of states. On the one hand, the U-*intersection* submodel defined in [21] has U as its domain, with its neighbourhood function built by restricting each set in a neighbourhood to the new domain, analogous to what $\mathcal{M}_\chi$ (a standard announcement) does. On the other hand, the U*-subset* submodel therein also has U as its domain, but its neighbourhood function is built by keeping only those sets that are already a subset of the new domain, analogous to what $\mathcal{M}_{!\chi}$ does. We argue that this second approach is more appropriate in the context of knowing how.

Even with this, more restricted, version of update, the resulting logic fails to have reduction axioms as the following proposition shows.

Proposition 2. $\mathsf{PAL}_{\mathsf{Kh}_i}$ *is more expresive than* $\mathsf{L}_{\mathsf{Kh}_i}$ *over arbitrary* $\mathrm{LTS}^{\mathrm{U}}$*s.*

Proof. Let $\mathcal{M}$ and $\mathcal{M}'$ the single agent models depicted below (states and edges depicted with dashed lines are those removed in $\mathcal{M}_{!r}$ and $\mathcal{M}'_{!r}$, respectively), with $\mathbb{S}_i := \{\{ab\}\}$ and $\mathbb{S}'_i := \{\{a\}\}$:



RAtom	$\vdash [!\chi]p \leftrightarrow (\chi \rightarrow p)$
R¬	$\vdash [!\chi]\neg\varphi \leftrightarrow (\chi \rightarrow \neg[!\chi]\varphi)$
R∨	$\vdash [!\chi](\varphi \vee \psi) \leftrightarrow [!\chi]\varphi \vee [!\chi]\psi$
RKh	$\vdash [!\chi]\mathsf{Kh}_i(\varphi, \psi) \leftrightarrow (\chi \rightarrow \mathsf{Kh}_i(\chi \wedge [!\chi]\varphi, \chi \wedge [!\chi]\psi))$
$\mathsf{RE}_{[!]}$	From $\vdash \varphi \leftrightarrow \psi$ derive $\vdash [!\chi]\varphi \leftrightarrow [!\chi]\psi$

Table 2: Reduction axioms $\mathcal{L}_{\mathsf{PAL}_{\mathsf{Kh}_i}}$.

Both models are $\mathsf{L}_{\mathsf{Kh}_i}$-bisimilar (Def. 8); hence, they satisfy the same formulas in $\mathsf{L}_{\mathsf{Kh}_i}$. However, $\mathcal{M}, w \not\models [!r]\mathsf{Kh}_i(p, q)$ since $\mathcal{M}, w \models r$ and $\mathcal{M}_{!r}, w \not\models \mathsf{Kh}_i(p, q)$, whereas $\mathcal{M}', w' \models [!r]\mathsf{Kh}_i(p, q)$ since $\mathcal{M}', w' \models r$ and $\mathcal{M}'_{!r}, w \models \mathsf{Kh}_i(p, q)$.

By furthermore restricting the class of models in which we will evaluate formulas, we are able to obtain reasonable reduction axioms.

Note that $\mathrm{LTS}^{\mathrm{U}}$s contain a set $\mathbb{S}_i$ of sets of plans for each agent i, which determines the perception of the agent w.r.t. her abilities. For instance, it may be the case that two plans ab and cd belong to some $\pi \in \mathbb{S}_i$, i.e., they are indistinguishable for agent i. In [5] it has been shown that the logic cannot distinguish between the class of arbitrary $\mathrm{LTS}^{\mathrm{U}}$s, and the class of models where each $\pi \in \mathbb{S}_i$ is a singleton with $\pi \subseteq \mathsf{Act}$. This is no longer the case in the presence of $[!\chi]$ (as the proof of Prop. 2 shows).

Definition 11. *Define* $\mathbf{M}^1$ *as the class of models* $\mathcal{M} = \langle \mathrm{W}, \mathrm{R}, \mathbb{S}, \mathrm{V} \rangle$ *such that for all* $i \in \mathsf{Agt}$ *and* $\pi \in \mathbb{S}_i$, $\pi \subseteq \mathsf{Act}$*.*

$\mathbf{M}^1$ constitutes a restricted class of models, which could correspond, for example, to a more abstract representation of the abilities of the agents, in which a course of action is modeled as a single action. The reduction axioms from Table 2 are valid in the class of models $\mathbf{M}^1$. Moreover, we can use them to eliminate announcements by iteratively replacing the innermost occurrence of a $[!\chi]$ modality. Thus, we get completeness for $\mathsf{PAL}_{\mathsf{Kh}_i}$.

Theorem 3. $\mathcal{L}_{\mathsf{Kh}_i}$ *together with the reduction axioms for* $[!\chi]$ *in Table 2 are a sound and strongly complete axiomatization for* $\mathsf{PAL}_{\mathsf{Kh}_i}$ *w.r.t.* $\mathbf{M}^1$*.*

3.2 Ontic Updates via Arrow Updates

Another framework for modifying relational models is *Arrow Update Logic* (AUL; [17]). It differs from PAL in that it removes only *edges*, thus keeping the domain intact. In standard epistemic logic, this corresponds to changes in uncertainty (e.g., the epistemic indistinguishability might be reduced, so intuitively the agents gain knowledge). For knowing how logics, the situation is different: updating edges in an LTS corresponds to updating the abilities of the agents, as arrows represent execution of actions. We introduce now a logic for arrow updates in the context of our knowing how logic.

Definition 12. *Formulas of the language* $\mathsf{AUL}_{\mathsf{Kh}_i}$ *are given by*

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \mathsf{Kh}_i(\varphi, \varphi) \mid [U]\varphi,$$
$$U ::= (\varphi, \varphi) \mid U, (\varphi, \varphi),$$

with $p \in \mathsf{Prop}$ *and* $i \in \mathsf{Agt}$*.*

RJoin	$[U]\varphi \leftrightarrow [(\bigwedge_{i=1}^{n} \theta_i, \bigwedge_{i=1}^{n} \theta'_i)]\varphi$
RAtom	$[(\theta,\theta')]p \leftrightarrow p$
R$\neg$	$[(\theta,\theta')]\neg\varphi \leftrightarrow \neg[(\theta,\theta')]\varphi$
R$\vee$	$[(\theta,\theta')](\varphi \vee \psi) \leftrightarrow [(\theta,\theta')]\varphi \vee [(\theta,\theta')]\psi$
RKh	$[(\theta,\theta')]\mathsf{Kh}_i(\varphi,\psi) \leftrightarrow \mathsf{A}([(\theta,\theta')]\varphi \to \theta) \wedge \mathsf{Kh}_i([(\theta,\theta')]\varphi, \theta' \wedge [(\theta,\theta')]\psi)$
RE$_U$	From $\vdash \varphi \leftrightarrow \psi$ derive $\vdash [(\theta,\theta')]\varphi \leftrightarrow [(\theta,\theta')]\psi$

Table 3: Reduction axioms $\mathcal{L}_{\mathsf{AUL}_{\mathsf{Kh}_i}}$ with $U = (\theta_1,\theta'_1),\ldots,(\theta_n,\theta'_n)$.

Definition 13. *Let $\mathcal{M} = \langle \mathrm{W}, \mathrm{R}, \mathbb{S}, \mathrm{V}\rangle$ be an $\mathrm{LTS}^{\mathrm{U}}$, and $U = (\theta_1,\theta'_1),\ldots,(\theta_n,\theta'_n)$ be such that θ_i, θ'_i are $\mathsf{AUL}_{\mathsf{Kh}_i}$-formulas, for all $0 \leq i \leq n$. We define $\mathcal{M}_U = \langle \mathrm{W}, \mathrm{R}_U, \mathbb{S}, \mathrm{V}\rangle$, where for every $a \in \mathsf{Act}$,*

$$(\mathrm{R}_U)_a = \{(w,v) \in \mathrm{R}_a(w) \mid w \in [\![\bigwedge_{i=1}^{n} \theta_i]\!]^{\mathcal{M}},\ \mathrm{R}_a(w) \subseteq [\![\bigwedge_{i=1}^{n} \theta'_i]\!]^{\mathcal{M}}\}.$$

Note that if $w \in [\![\bigwedge_{i=1}^{n} \theta_i]\!]^{\mathcal{M}}$ and $\mathrm{R}_a(w) \subseteq [\![\bigwedge_{i=1}^{n} \theta'_i]\!]^{\mathcal{M}}$, then $\mathrm{R}'_a(w) = \mathrm{R}_a(w)$. Moreover, $\mathrm{R}'_a(w) \neq \emptyset$ iff $w \in [\![\bigwedge_{i=1}^{n} \theta_i]\!]^{\mathcal{M}}$, $\mathrm{R}_a(w) \subseteq [\![\bigwedge_{i=1}^{n} \theta'_i]\!]^{\mathcal{M}}$ and $\mathrm{R}_a(w) \neq \emptyset$.

Once again, the update here differs from the original one in e.g., [17], in that given a state satisfying the precondition, it takes in consideration all the states that are reachable from it. Thus, the satisfaction of the postcondition at all those states defines whether the arrows are preserved or not.

Definition 14. *We extend the satisfaction relation $\models$ from Def. 6 with the case:*

$$\mathcal{M}, w \models [U]\varphi \text{ iff } \mathcal{M}_U, w \models \varphi.$$

As in the PAL case, AUL performs ontic updates rather than epistemic updates over $\mathrm{LTS}^{\mathrm{U}}$-based knowing how.

Proposition 3. *$\mathsf{AUL}_{\mathsf{Kh}_i}$ is more expressive than $\mathsf{L}_{\mathsf{Kh}_i}$ over arbitrary $\mathrm{LTS}^{\mathrm{U}}$s.*

Proof. By using the models from Prop. 2, we have that $\mathcal{M}, w \not\models [(r,r)]\mathsf{Kh}_i(p,q)$ and $\mathcal{M}', w' \models [(r,r)]\mathsf{Kh}_i(p,q)$.

Again, the reduction axioms from Table 3 are valid in the class of models $\mathbf{M}^1$, and we can use them to eliminate all the occurrences of the $[U]$ modality.

Theorem 4. *$\mathcal{L}_{\mathsf{Kh}_i}$ together with the reduction axioms for $[U]$ in Table 3 are a sound and strongly complete axiomatization for $\mathsf{AUL}_{\mathsf{Kh}_i}$ w.r.t. $\mathbf{M}^1$.*

3.3 Epistemic Updates, Preliminary Thoughts

In this section we present some preliminary results on different ways in which interesting *epistemic* updates can be introduced in the context of a knowing how operator. No complete axiomatization is available yet. Instead, we will discuss a number of proposals for update operators and show that they can be used to express some relevant properties.

Removing uncertainty between two plans. One of the advantages of $\mathrm{LTS}^{\mathrm{U}}$s is that they allow a natural representation of actions that affect the abilities of an agent, but also her epistemic state. In an $\mathrm{LTS}^{\mathrm{U}}$, the crucial epistemic component is the set $\mathbb{S}_i$, defining not only the plans agent i is 'aware of', but also the level at which she can discern among them. Thus we can represent changes in the epistemic state of an agent by means of operations that modify $\mathbb{S}_i$.

Example 2. Let $\mathcal{M}$ be the LTS$^{\mathrm{U}}$ from Ex. 1. Recall that $\mathcal{M} \not\models \mathsf{Kh}_j(h, g)$. The conflicting plan is *ebmfsp*, which does not lead to a good cake. Thus, if agent j is able to tell apart *ebmfsp* from *ebfmsp* (which is the good plan), she would be able to know how to get a good cake, provided she has the ingredients. If agent j *learns* that the order of the actions matters (so *ebmfsp* is distinct from *ebfmsp*), the set $\pi = \{ebfmsp, ebmfsp\}$ is split into two singleton sets. After such a splitting, she knows how to achieve g given h.

We introduce an operation that eliminates uncertainty between specific plans. In an LTS$^{\mathrm{U}}$, there might be different ways of making distinguishable two previously indistinguishable plans: the different ways one can split a set containing both. First, some notation.

Definition 15. *Let $\pi, \pi_1, \pi_2 \in 2^{\mathsf{Act}^*}$, and $S \subseteq 2^{\mathsf{Act}^*}$. We write $\pi = \pi_1 \uplus \pi_2$ iff $\pi = \pi_1 \cup \pi_2$ and $\pi_1 \cap \pi_2 = \emptyset$.*

For $\pi \in S$ and $\pi = \pi_1 \uplus \pi_2$, define $S^{\pi}_{\{\pi_1,\pi_2\}} \subseteq 2^{\mathsf{Act}^*}$ as the result of refining π through $\{\pi_1, \pi_2\}$: $S^{\pi}_{\{\pi_1,\pi_2\}} := (S \setminus \{\pi\}) \cup \{\pi_1, \pi_2\}$.

Definition 16. *Let $S, S' \subseteq 2^{(\mathsf{Act}^*)}$; and let $\sigma_1, \sigma_2 \in \mathsf{Act}^*$ be such that $\sigma_1 \neq \sigma_2$. We write $S \rightsquigarrow^{\sigma_1}_{\sigma_2} S'$ if and only if either*

- *$S' = S$ and there is no $\pi \in S$ satisfying $\{\sigma_1, \sigma_2\} \subseteq \pi$, or*
- *$S' = S^{\pi}_{\{\pi_1,\pi_2\}}$ for some $\pi \in S$ satisfying $\{\sigma_1, \sigma_2\} \subseteq \pi$, with $\pi_1, \pi_2 \in 2^{\mathsf{Act}^*}$ such that $\pi = \pi_1 \uplus \pi_2$ and $\sigma_1 \in \pi_1$, $\sigma_2 \in \pi_2$.*

Note: the relation $\rightsquigarrow^{\sigma_1}_{\sigma_2}$ is serial and functional. Moreover, if S is the set of sets of plans for a given agent i in some LTS$^{\mathrm{U}}$ (i.e., $S = \mathbb{S}_i$) and S' is the unique set satisfying $S \rightsquigarrow^{\sigma_1}_{\sigma_2} S'$, then the structure resulting from replacing S by S' is an LTS$^{\mathrm{U}}$.

Definition 17. *Let $\mathcal{M} = \langle \mathrm{W}, \mathrm{R}, \mathbb{S}, \mathrm{V} \rangle$ be an LTS$^{\mathrm{U}}$, and let $\mathbb{S}' = \{\mathbb{S}'_i\}_{i \in \mathsf{Agt}}$ with $\mathbb{S}'_i \subseteq 2^{(\mathsf{Act}^*)}$. Let $\sigma_1, \sigma_2 \in \mathsf{Act}^*$. We write $\mathbb{S} \rightsquigarrow^{\sigma_1}_{\sigma_2} \mathbb{S}'$ iff for each $i \in \mathsf{Agt}$, $\mathbb{S}_i \rightsquigarrow^{\sigma_1}_{\sigma_2} \mathbb{S}'_i$. We denote by $\mathcal{M}^{\mathbb{S}}_{\mathbb{S}'}$ the LTS$^{\mathrm{U}}$ obtained by replacing $\mathbb{S}$ by $\mathbb{S}'$.*

The definition above guarantees there is a one-to-one correspondence between the sets in $\mathbb{S}$ and those in $\mathbb{S}'$. With these tools at hand, we introduce the new modality $\langle \sigma_1 \not\sim \sigma_2 \rangle$, semantically interpreted as an action through which all agents learn that plans σ_1 and σ_2 are different. We use $\mathsf{L}_{\mathsf{Ref}}$ (Ref for "refinement") to denote the extension of $\mathsf{L}_{\mathsf{Kh}_i}$ with $\langle \sigma_1 \not\sim \sigma_2 \rangle$.

Definition 18. *Let $\mathcal{M} = \langle \mathrm{W}, \mathrm{R}, \mathbb{S}, \mathrm{V} \rangle$ be an LTS$^{\mathrm{U}}$ and $w \in \mathrm{W}$. For $\sigma_1 \neq \sigma_2$,*

$$\mathcal{M}, w \models \langle \sigma_1 \not\sim \sigma_2 \rangle \varphi \quad \mathit{iff}_{\mathit{def}} \quad \text{there is } \mathbb{S}' \text{ s.t. } \mathbb{S} \rightsquigarrow^{\sigma_1}_{\sigma_2} \mathbb{S}' \text{ and } \mathcal{M}^{\mathbb{S}}_{\mathbb{S}'}, w \models \varphi.$$

As usual, we define $[\sigma_1 \not\sim \sigma_2]\varphi := \neg\langle \sigma_1 \not\sim \sigma_2 \rangle \neg\varphi$.

Formulas of the form $\langle \sigma_1 \not\sim \sigma_2 \rangle \varphi$ can be read as follows: *"after it is stated that plans σ_1 and σ_2 are distinguishable, φ holds"*. For instance, taking Ex. 2, $\langle ebmfsp \not\sim ebfmsp \rangle \mathsf{Kh}_j(h, g)$, establishes that *"after it is stated that ebmfsp and ebfmsp are distinguishable plans, agent j knows how to produce a good cake, provided she has the ingredientes"*.

The proposed modality has some natural properties: it is normal and serial.

Proposition 4. *It follows from the semantics that:*

1. $\models [\sigma_1 \not\sim \sigma_2](\varphi \to \psi) \to ([\sigma_1 \not\sim \sigma_2]\varphi \to [\sigma_1 \not\sim \sigma_2]\psi)$.
2. *If* $\models \varphi$*, then* $\models [\sigma_1 \not\sim \sigma_2]\varphi$.
3. $\models [\sigma_1 \not\sim \sigma_2]\varphi \to \langle \sigma_1 \not\sim \sigma_2 \rangle \varphi$.

This dynamic modality both preserves knowledge and can generate new one.

Proposition 5. *Let* φ, ψ *be* propositional *formulas. Then,*

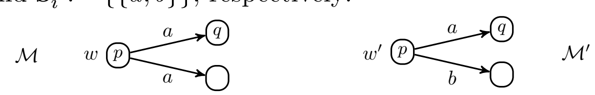
1. $\models \mathsf{Kh}_i(\varphi, \psi) \to [\sigma_1 \not\sim \sigma_2]\mathsf{Kh}_i(\varphi, \psi)$.
2. $\neg\mathsf{Kh}_i(\varphi, \psi) \wedge [\sigma_1 \not\sim \sigma_2]\mathsf{Kh}_i(\varphi, \psi)$ *is satisfiable.*

Proof. For Item 1, suppose $\mathcal{M}, w \models \mathsf{Kh}_i(\varphi, \psi)$. Then there is $\pi \in \mathbb{S}_i$ s.t. $[\![\varphi]\!]^{\mathcal{M}} \subseteq \mathrm{SE}(\pi)$ and $\mathrm{R}_\pi([\![\varphi]\!]^{\mathcal{M}}) \subseteq [\![\psi]\!]^{\mathcal{M}}$. Let $\sigma_1, \sigma_2 \in \mathsf{Act}^*$. If $\sigma_1 \notin \pi$ or $\sigma_2 \notin \pi$, then π does not change and is still the witness for $\mathsf{Kh}_i(\varphi, \psi)$. If, however, $\sigma_1, \sigma_2 \in \pi$, there will be a partition of π, $\{\pi_1, \pi_2\}$ s.t. $\mathbb{S}_i \rightsquigarrow^{\sigma_1}_{\sigma_2} \mathbb{S}_i{}^{\pi}_{\{\pi_1,\pi_2\}}$. But this does not cause any problem since $[\![\varphi]\!]^{\mathcal{M}} \subseteq \mathrm{SE}(\pi) \subseteq \mathrm{SE}(\pi_k)$ and $\mathrm{R}_{\pi_k}([\![\varphi]\!]^{\mathcal{M}}) \subseteq \mathrm{R}_\pi([\![\varphi]\!]^{\mathcal{M}}) \subseteq [\![\psi]\!]^{\mathcal{M}}$, for $k \in \{1, 2\}$. Here agent i knew how to go from φ-states to ψ-states via π. Weakening such π by making a partition still holds the property, allowing the agent to choose between π_1 or π_2 as her next witness. Since all the cases for σ_1 and σ_2 are covered, $\mathcal{M}, w \models [\sigma_1 \not\sim \sigma_2]\mathsf{Kh}_i(\varphi, \psi)$. For Item 2, see Ex. 2.

The new modality adds expressivity, as it can talk explicitly about plans:

Proposition 6. $\mathsf{L}_{\mathsf{Ref}}$ *is more expressive than* $\mathsf{L}_{\mathsf{Kh}_i}$.

Proof. We need to display two $\mathsf{L}_{\mathsf{Kh}_i}$-bisimilar $\mathrm{LTS}^{\mathrm{U}}$s that can be distinguished by an $\mathsf{L}_{\mathsf{Ref}}$-formula. Let $\mathcal{M}$ and $\mathcal{M}'$ the single agent models depicted below, with $\mathbb{S}_i := \{\{a\}\}$ and $\mathbb{S}'_i := \{\{a, b\}\}$, respectively:



The models are $\mathsf{L}_{\mathsf{Kh}_i}$-bisimilar, thus they satisfy the same formulas in $\mathsf{L}_{\mathsf{Kh}_i}$ (in particular $\neg\mathsf{Kh}_i(p, q)$). But, $\mathcal{M}, w \not\models \langle a \not\sim b \rangle \mathsf{Kh}_i(p, q)$ since $\mathbb{S} \rightsquigarrow^a_b \mathbb{S}$, whereas $\mathcal{M}', w' \models \langle a \not\sim b \rangle \mathsf{Kh}_i(p, q)$, since there is $\mathbb{S}'' = \{\{a\}, \{b\}\}$ s.t. $\mathbb{S}' \rightsquigarrow^a_b \mathbb{S}''$.

Arbitrary refinement over plans. As mentioned, the operation $\langle \sigma_1 \not\sim \sigma_2 \rangle$ can be seen as a particular form of (publicly) removing uncertainty: one indicates precisely the plans that can be distinguished now, and then quantifies over the different ways of doing so. The operation defined below is a more abstract one: in the spirit of other proposals that quantify over epistemic actions (e.g., the arbitrary announcements of [6], the arbitrary arrow updates of [29], the group announcements of [1] and the coalition announcements of [3]), it quantifies over all the different ways in which the agent's indistinguishability can be refined.

Definition 19. *Let* $\mathcal{M}$ *be an* $\mathrm{LTS}^{\mathrm{U}}$ *and* $w \in \mathrm{D}_{\mathcal{M}}$. *Then,*

$$\mathcal{M}, w \models \langle \not\sim \rangle \varphi \text{ iff}_{def} \text{ there are } \sigma_1, \sigma_2 \in \mathsf{Act}^* \text{ s.t. } \mathcal{M}, w \models \langle \sigma_1 \not\sim \sigma_2 \rangle \varphi.$$

As usual $[\not\sim]\varphi = \neg\langle \not\sim \rangle \neg\varphi$. *We denote* $\mathsf{L}_{\mathsf{ARef}}$ *(for "arbitrary refinement") as the extension of* $\mathsf{L}_{\mathsf{Kh}_i}$ *with the modality* $\langle \not\sim \rangle$.

The resulting modality is normal and serial, satisfies natural properties of Monotonicity and Weakening, but fails for dynamic versions of axioms 4 and 5.

Proposition 7. *It follows from the semantics that:*

1. $\models [\not\sim](\varphi \to \psi) \to ([\not\sim]\varphi \to [\not\sim]\psi)$.
2. *If* $\models \varphi$, *then* $\models [\not\sim]\varphi$.
3. $\models [\not\sim]\varphi \to \langle\not\sim\rangle\varphi$.
4. $\models \langle\not\sim\rangle\varphi \to \langle\not\sim\rangle(\varphi \vee \psi)$ *and* $\models [\not\sim]\varphi \to [\not\sim](\varphi \vee \psi)$ *(Monotonicity).*
5. $\models \langle\not\sim\rangle(\varphi \wedge \psi) \to \langle\not\sim\rangle\varphi$ *and* $\models [\not\sim](\varphi \wedge \psi) \to [\not\sim]\varphi$ *(Weakening).*
6. $\not\models [\not\sim]\varphi \to [\not\sim][\not\sim]\varphi$ *(axiom 4).*
7. $\not\models \neg[\not\sim]\varphi \to [\not\sim]\neg[\not\sim]\varphi$ *(axiom 5).*

By definition, $\models \langle\sigma_1 \not\sim \sigma_2\rangle\varphi \to \langle\not\sim\rangle\varphi$, but the exact expressivity relation between the two resulting logics requires further developments. In particular, given the mismatch between the two languages ($\mathsf{L}_{\mathsf{Ref}}$ is able to talk about specific plans whereas $\mathsf{L}_{\mathsf{ARef}}$ is not), it does not seem trivial to give a translation from one logic to the other. However, by using the same argument as in Prop. 6, it is easy to show the following:

Proposition 8. $\mathsf{L}_{\mathsf{ARef}}$ *is more expressive than* $\mathsf{L}_{\mathsf{Kh}_i}$.

Goal directed learning how. One might notice that knowing how operators are *goal-directed*: the agent looks for a suitable course of action that makes her achieve a certain state. It is possible to define an operator that, when possible, *guarantees* that the agent *learns how* to achieve a goal. This action can be understood as a goal-directed learning how: it looks for a way to split *some* existing set of plans π in such a way that the agent knows how to achieve φ given ψ.

Let L_{Lh} (for "learning how") be $\mathsf{L}_{\mathsf{Kh}_i}$ extended with the dynamic modality

$$\langle\psi, \varphi\rangle_i\chi := \langle\not\sim\rangle(\mathsf{Kh}_i(\psi, \varphi) \wedge \chi),$$

(and its 'dual' $[\psi, \varphi]_i\chi := \neg\langle\psi, \varphi\rangle_i\neg\chi$). Moreover, we define $\mathsf{L}_i(\psi, \varphi) := \langle\psi, \varphi\rangle_i\top$ an abbreviation for *"the agent i can learn how to make φ true in the presence of ψ"*. Notice that L_{Lh} is a syntactic fragment of $\mathsf{L}_{\mathsf{ARef}}$.

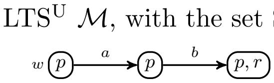
The new dynamic modality is a ternary modality expressing that the agent is able to learn how to achieve φ given ψ, and that after this learning operation takes place, χ holds. The modality L_i is a test of what is learnable by the agent i. The next proposition states some interesting properties of these modalities.

Proposition 9. *It follows from the semantics that:*

1. $\not\models \mathsf{L}_i(\varphi, \psi)$*;*
2. $\mathsf{L}_i(\varphi, \psi) \wedge \mathsf{L}_i(\varphi, \neg\psi)$ *is satisfiable.*

Proof. Item 1 shows that not everything is learnable by an agent. The (un)availability of certain actions in an $\mathrm{LTS}^{\mathrm{U}}$ restricts what can be learnt. Consider the following single-agent $\mathrm{LTS}^{\mathrm{U}}$ $\mathcal{M}$, with the set $\mathbb{S}_i$ shown on the right.

$\mathcal{M}$



$\mathbb{S}_i = \{\, \{ab, a\}, \{\epsilon\} \,\}$

Note that $\mathcal{M}, w \not\models \mathsf{Kh}_i(p, r)$. The set $\{ab, a\}$ is not executable at every p-state, it is only executable at w. On the other hand, $\{\epsilon\}$ is executable everywhere, but does not lead always to r-states. Moreover, $\mathcal{M}, w \not\models \mathsf{L}_i(p, r)$. The set $\{\epsilon\}$ cannot be refined, and no refinement of $\{ab, a\}$ does the work. Therefore, agent i cannot learn how to make r true when p holds.

For Item 2 consider the model $\mathcal{M}'$ in Prop. 6. As said, $\mathcal{M}', w' \not\models \mathsf{Kh}_i(p, q)$. However, there is a way to learn how to achieve q given p: it is possible to split the set $\{a, b\}$ into $\{a\}$ and $\{b\}$; hence, $\mathcal{M}', w' \models \mathsf{L}_i(p, q)$ (witness $\{a\}$) but also $\mathcal{M}', w' \models \mathsf{L}_i(p, \neg q)$ (witness $\{b\}$).

Item 1 shows how, in certain scenarios, there is no room for learning. For instance, there might be no way to learn how to cure a disease, if there is no doctor available. Item 2 shows how the agent might be able to learn not only how to make a formula true under a given condition, but, at the same time, how to make the same formula false under the same condition.

Once more, $[\chi, \psi]$ (seen as a unary modality) is a *normal modality*:

Proposition 10. *The modality* $[\chi, \psi]$ *is normal:*

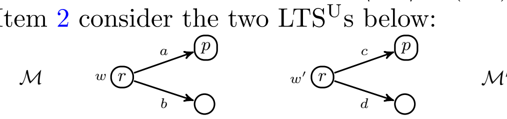
1. $\models [\chi, \psi](\theta \to \varphi) \to ([\chi, \psi]\theta \to [\chi, \psi]\varphi)$.
2. *If* $\models \varphi$, *then* $\models [\chi, \psi]\varphi$.

We finish the section by stating some expressivity connections between the dynamic modalities we just discussed.

Proposition 11. *The following propositions are true:*

1. L_{Lh} *is more expressive than* $\mathsf{L}_{\mathsf{Kh}_i}$.
2. L_{Lh} *is not more expressive than* $\mathsf{L}_{\mathsf{Ref}}$.

Proof. Item 1 is proved as Prop. 6: the formula $\langle p, q\rangle \mathsf{Kh}(p, q)$ distinghuishes the two LTS^{U}s. For Item 2 consider the two LTS^{U}s below:



For each model, consider respective sets $\mathbb{S}_i = \{\{a, b\}\}$ and $\mathbb{S}'_i = \{\{c, d\}\}$. Since L_{Lh} cannot explicitely talk about plans, $\mathcal{M}, w$ and $\mathcal{M}', w'$ are indistinguishable for it. In $\mathsf{L}_{\mathsf{Ref}}$, $\mathcal{M}, w \models \langle a \not\sim b\rangle \mathsf{Kh}_i(r, p)$ and $\mathcal{M}', w' \not\models \langle a \not\sim b\rangle \mathsf{Kh}_i(r, p)$.

4 Conclusions

Taking the uncertainty-based semantics from [5] as our starting point, we investigated dynamic modalities in the context of *knowing how* logics. In this regard, we studied two forms of updates: ontic updates, via annoucement-like and arrow-update-like modalities; and epistemic updates, refining the perception of an agent regarding her own abilities. For the operators encompassed in the former family, we provided axiomatizations over a particular class of models, via reductions axioms; for the latter family, we discussed some preliminary thoughts and semantic properties of each operator.

We consider this to be the first step towards a more general theory of dynamic epistemic logics for knowing how. Moreover, our work opens the path to study other dynamic operators in this context. For instance, it is known that dynamic operators do not satisfy uniform substitution in general (see, e.g., [4]). It would be interesting to explore alternative techniques for obtaining proof systems without a general rule of substitution. Another approach could be playing with the operators' expressivity (e.g., by expressing other properties about the abilities), in order to find fragments that are axiomatizable via reduction axioms.

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References

1. T. Ågotnes, P. Balbiani, H. van Ditmarsch, and P. Seban. Group announcement logic. *Journal of Applied Logic*, 8(1):62–81, 2009.
2. T. Ågotnes, V. Goranko, W. Jamroga, and M. Wooldridge. Knowledge and ability. In H. van Ditmarsch, J. Y. Halpern, W. van der Hoek, and B. Kooi, editors, *Handbook of Epistemic Logic*, chapter 11, pages 543–589. College Pub., 2015.
3. T. Ågotnes and H. van Ditmarsch. Coalitions and announcements. In *7th International Joint Conference on Autonomous Agents and Multiagent Systems (AAMAS 2008), Volume 2*, pages 673–680. IFAAMAS, 2008.
4. C. Areces, R. Fervari, and G. Hoffmann. Relation-changing modal operators. *Logic Journal of the IGPL*, 23(4):601–627, 2015.
5. C. Areces, R. Fervari, A. R. Saravia, and F. R. Velázquez-Quesada. Uncertainty-based semantics for multi-agent knowing how logics. In *Proceedings Eighteenth Conference on Theoretical Aspects of Rationality and Knowledge, TARK 2021*, volume 335 of *EPTCS*, pages 23–37, 2021.
6. P. Balbiani, A. Baltag, H. van Ditmarsch, A. Herzig, T. Hoshi, and T. de Lima. 'Knowable' as 'known after an announcement'. *Review of Symbolic Logic*, 1(3):305–334, 2008.
7. F. Belardinelli. Reasoning about knowledge and strategies: Epistemic strategy logic. In *Proceedings 2nd International Workshop on Strategic Reasoning, SR 2014*, volume 146 of *EPTCS*, pages 27–33, 2014.
8. J. Fantl. Knowledge how. In *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, spring 2021 edition, 2021.
9. R. Fervari, A. Herzig, Y. Li, and Y. Wang. Strategically knowing how. In *Proceedings of the Twenty-Sixth International Joint Conference on Artificial Intelligence, IJCAI 2017*, pages 1031–1038. ijcai.org, 2017.
10. R. Fervari, F. R. Velázquez-Quesada, and Y. Wang. Bisimulations for knowing how logics. *The Review of Symbolic Logic*, 15(2):450–486, 2022.
11. V. Goranko and S. Passy. Using the universal modality: Gains and questions. *Journal of Logic and Computation*, 2(1):5–30, 1992.
12. S. O. Hansson. Logic of Belief Revision. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Spring 2022 edition, 2022.

13. A. Herzig. Logics of knowledge and action: critical analysis and challenges. *Autonomous Agents and Multi Agent Systems*, 29(5):719–753, 2015.
14. A. Herzig and N. Troquard. Knowing how to play: uniform choices in logics of agency. In *5th International Joint Conference on Autonomous Agents and Multiagent Systems (AAMAS 2006)*, pages 209–216. ACM, 2006.
15. W. Jamroga and T. Ågotnes. Constructive knowledge: what agents can achieve under imperfect information. *Journal of Applied Non Classical Logics*, 17(4):423–475, 2007.
16. W. Jamroga and W. van der Hoek. Agents that know how to play. *Fundamenta Informaticae*, 63(2-3):185–219, 2004.
17. B. Kooi and B. Renne. Arrow update logic. *Review of Symbolic Logic*, 4(4):536–559, 2011.
18. Y. Lespérance, H. J. Levesque, F. Lin, and R. B. Scherl. Ability and knowing how in the situation calculus. *Studia Logica*, 66(1):165–186, 2000.
19. Y. Li. Stopping means achieving: A weaker logic of knowing how. *Studies in Logic*, 9(4):34–54, 2017.
20. Y. Li and Y. Wang. Achieving while maintaining: - A logic of knowing how with intermediate constraints. In *Logic and Its Applications*, volume 10119 of *LNCS*, pages 154–167. Springer, 2017.
21. M. Ma and K. Sano. How to update neighbourhood models. *Journal of Logic and Computation*, 28(8):1781–1804, 2018.
22. J. McCarthy and P. J. Hayes. Some philosophical problems from the standpoint of artificial intelligence. In *Machine Intelligence*, pages 463–502. Edinburgh University Press, 1969.
23. R. Montague. Universal grammar. *Theoria*, 36(3):373–398, 1970.
24. J. Plaza. Logics of public communications. In *Proceedings of the 4th International Symposium on Methodologies for Intelligent Systems*, pages 201–216, 1989.
25. D. Scott. Advice on modal logic. In Karel Lambert, editor, *Philosophical Problems in Logic*, pages 143–173. Reidel, Dordrecht, The Netherlands, 1970.
26. W. van der Hoek and A. Lomuscio. Ignore at your peril - towards a logic for ignorance. In *The Second International Joint Conference on Autonomous Agents & Multiagent Systems, AAMAS 2003, Proceedings*, pages 1148–1149. ACM, 2003.
27. H. van Ditmarsch and B. Kooi. Semantic results for ontic and epistemic change. In G. Bonanno, W. van der Hoek, and M. Wooldridge, editors, *Logic and the Foundations of Game and Decision Theory*, pages 87–117. Amsterdam University Press, 2008.
28. H. van Ditmarsch, W. van der Hoek, and B. Kooi. *Dynamic Epistemic Logic*. Springer, 2007.
29. H. van Ditmarsch, W. van der Hoek, B. Kooi, and L. B. Kuijer. Arbitrary arrow update logic. *Artificial Intelligence*, 242:80–106, 2017.
30. X. Wang. A logic of knowing how with skippable plans. In *Logic, Rationality, and Interaction - 7th International Workshop, LORI 2019, Proceedings*, volume 11813 of *LNCS*, pages 413–424. Springer, 2019.
31. Y. Wang. A logic of knowing how. In *Logic, Rationality, and Interaction - 5th International Workshop, LORI 2015, Proceedings*, volume 9394 of *LNCS*, pages 392–405. Springer, 2015.
32. Y. Wang. A logic of goal-directed knowing how. *Synthese*, 195(10):4419–4439, 2018.

Parametrized modal logic II: the unidimensional case

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Abstract. We consider a syntax and semantics of modal logics based on parametrized modal connectives with $\exists\forall$-satisfaction definitions, we axiomatically introduce different parametrized modal logics, we prove their completeness with respect to appropriate classes of parametrized relational structures and we show the decidability of some related satisfiability problems.

Keywords: Parametrized modal logic. Completeness. Decidability.

1 Introduction

The connective $\Diamond$ of arity 1 usually considered in the propositional modal language has a $\exists$-satisfaction definition: in relational models of the form (W, R, V) where R is a binary relation on a nonempty set W of possible worlds, V interprets formulas in such a way that for all formulas φ, the possible world s is in $V(\Diamond\varphi)$ exactly when for some possible world t, sRt and $t \in V(\varphi)$. Within the context of temporal reasoning, the until connective $\mathcal{U}$ of arity 2 has been considered in order to increase the expressive power of the propositional modal language, its $\exists\forall$-satisfaction definition being such that in models (W, R, V) as above, for all formulas φ, ψ, the possible world s is in $V(\varphi\mathcal{U}\psi)$ exactly when for some possible world u, sRu, $u \in V(\psi)$ and for every possible world t, if sRt and tRu then $t \in V(\varphi)$. As is well-known, the use of the until connective $\mathcal{U}$ of arity 2 allows to characterize more classes of relational structures than we can characterize in the propositional modal language based on the connective $\Diamond$ of arity 1 [3, Chapter 7]. Moreover, the use of the until connective $\mathcal{U}$ has no dramatic consequence on the computational complexity of the satisfiability problem, this problem being $\mathbf{PSPACE}$-complete in the most popular classes of models usually considered for applications of temporal reasoning [16, 17].

Therefore, a question naturally arises: without dramatically affecting the computational complexity of the satisfiability problem, are there other ways to increase the expressive power of propositional modal languages by considering other connectives with complex satisfaction definitions? Let us consider a propositional modal language based on a connective $\Diamond$ of arity 2. Traditionally, its relational models are of the form (W, R, V) where R is a ternary relation on a nonempty set W of possible worlds and V interprets formulas in such a way that for all formulas φ, ψ, the possible world s is in $V(\varphi\Diamond\psi)$ exactly when for some possible world u, $u \in V(\psi)$ and for some possible world t, $t \in V(\varphi)$ and $sR(t, u)$. On the pattern of the until connective $\mathcal{U}$ and its $\exists\forall$-satisfaction definition, let us consider a propositional modal language based on a new connective $\blacklozenge$ of arity 2 and such that in models (W, R, V) as above, for all formulas φ, ψ, the possible world s is in $V(\varphi \blacklozenge \psi)$ exactly when for some possible world u, $u \in V(\psi)$ and for every possible world t, if $t \in V(\varphi)$ then $sR(t, u)$. With such syntax and semantics at hand, can we characterize more classes of relational structures than we can characterize in the propositional modal language based on the connective $\Diamond$ of arity 2? And what is the price to pay in terms of the computational complexity of the satisfiability problem?

Obviously, given a ternary relation R on a nonempty set W, we can naturally consider the function $\mathbf{R} : \wp(W) \longrightarrow \wp(W \times W)$ such that for all subsets A of W and for every s, u in W, $s\mathbf{R}(A)u$ exactly when for every t in W, if $t \in A$ then $sR(t, u)$. Obviously, the main property of such function is that for all subsets A of W, $\mathbf{R}(A) = \bigcap\{\mathbf{R}(\{t\}) : t \in A\}$. Reciprocally, given a nonempty set W and a function $\mathbf{R} : \wp(W) \longrightarrow \wp(W \times W)$ satisfying the above property, we can naturally consider the ternary relation R on W such that for every s, t, u in W, $sR(t, u)$ exactly when $s\mathbf{R}(\{t\})u$. This suggests us to consider a propositional modal language based on a connective $\blacklozenge$ of arity 2 and such that in models of the form $(W, \mathbf{R}, V)$ where W is a nonempty set of possible worlds and $\mathbf{R} : \wp(W) \longrightarrow \wp(W \times W)$ is a function satisfying the above property, for all formulas φ, ψ, the possible world s is in $V(\varphi \blacklozenge \psi)$ exactly when for some possible world u, $u \in V(\psi)$ and $s\mathbf{R}(V(\varphi))u$. In this paper, with such syntax (Section 2) and semantics (Section 3) at hand, we axiomatically introduce different modal logics (Section 4), we prove their completeness with respect to appropriate classes of relational structures (Sections 5 and 6) and we show the decidability of some related satisfiability problems (Section 7).

2 Syntax

Let $\mathcal{P}$ be a countably infinite set (with typical members denoted p, q, etc). Members of $\mathcal{P}$ will be called *atomic formulas.* A *tip* is a set Σ of finite words over the alphabet $\mathcal{P} \cup \{\bot, \neg, \vee, \blacklozenge, (,)\}$ (with typical members denoted φ, ψ, etc). Let $\mathcal{L}$ be the least tip such that $\mathcal{P} \subseteq \mathcal{L}$ and for all finite words φ, ψ,

- $\bot \in \mathcal{L}$,
- if $\varphi \in \mathcal{L}$ then $\neg\varphi \in \mathcal{L}$,
- if $\varphi, \psi \in \mathcal{L}$ then $(\varphi \vee \psi) \in \mathcal{L}$,
- if $\varphi, \psi \in \mathcal{L}$ then $(\varphi \blacklozenge \psi) \in \mathcal{L}$.

Members of $\mathcal{L}$ will be called *formulas.* The Boolean connectives $\top$, $\wedge$, $\rightarrow$ and $\leftrightarrow$ are defined as the usual abbreviations. For all $\varphi, \psi \in \mathcal{L}$, anticipating the fact that the roles of φ and ψ in $(\varphi \blacklozenge \psi)$ are not symmetric, let $(\varphi \blacksquare \psi)$ be an abbreviation of $\neg(\varphi \blacklozenge \neg\psi)$. We adopt the standard rules for omission of the parentheses. A tip Σ is *readable* if $\Sigma \subseteq \mathcal{L}$. A readable tip Σ is *closed* if for all $\varphi, \psi \in \mathcal{L}$,

- if $\neg\varphi \in \Sigma$ then $\varphi \in \Sigma$,
- if $\varphi \vee \psi \in \Sigma$ then $\varphi, \psi \in \Sigma$,
- if $\varphi \blacklozenge \psi \in \Sigma$ then $\varphi, \psi \in \Sigma$.

For all $\varphi \in \mathcal{L}$, let Σ_φ be the least closed readable tip containing φ. For all $\varphi \in \mathcal{L}$, let $\|\varphi\|$ be the length of φ.

Lemma 1. *For all $\varphi \in \mathcal{L}$, $\mathtt{Card}(\Sigma_\varphi) \leq \|\varphi\|$.*

From now on in this paper, for all $\varphi, \psi \in \mathcal{L}$, we will write "$\langle\varphi\rangle\psi$" instead of "$\varphi\blacklozenge\psi$" and "$[\varphi]\psi$" instead of "$\varphi\blacksquare\psi$". For all $\varphi \in \mathcal{L}$ and for all readable tips Σ, let $[\varphi]\Sigma$ be the set of all $\psi \in \mathcal{L}$ such that $[\varphi]\psi \in \Sigma$.

3 Relational semantics

A *frame* is a couple (W, R) where W is a nonempty set and $R : \wp(W) \longrightarrow \wp(W \times W)$. A frame (W, R) is *conjunctive* if for all $A \in \wp(W)$, $R(A) = \bigcap\{R(\{s\}) : s \in A\}$. A frame (W, R) is *preconjunctive* if $R(\emptyset) = W \times W$ and for all $A, B \in \wp(W)$, $R(A \cup B) = R(A) \cap R(B)$. A frame (W, R) is *paraconjunctive* if $R(\emptyset) = W \times W$ and for all $A, B \in \wp(W)$, if $A \subseteq B$ then $R(A) \supseteq R(B)$. A *frame of indiscernibility* is a frame (W, R) such that for all $A \in \wp(W)$, $R(A)$ is an equivalence relation on W.

Lemma 2. *Every conjunctive frame is preconjunctive.*

Example 1. There exist preconjunctive nonconjunctive frames. For instance, the frame (W, R) where $W = \mathbb{N}$ and for all $A \in \wp(\mathbb{N})$, if A is finite then $R(A) = \mathbb{N} \times \mathbb{N}$ else $R(A) = \emptyset$. Obviously, this frame is preconjunctive. However, it is not conjunctive, seeing that $R(\mathbb{N}) = \emptyset$ and $\bigcap\{R(\{s\}) : s \in \mathbb{N}\} = \mathbb{N} \times \mathbb{N}$.

Lemma 3. *Every preconjunctive frame is paraconjunctive.*

Example 2. There exist paraconjunctive nonpreconjunctive frames. For instance, the frame (W, R) where $W = \mathbb{N}$ and for all $A \in \wp(\mathbb{N})$, if $\mathtt{Card}(A) < 2$ then $R(A) = \mathbb{N} \times \mathbb{N}$ else $R(A) = \emptyset$. Obviously, this frame is paraconjunctive. However, it is not preconjunctive, seeing that $R(\{0, 1\}) = \emptyset$ and $R(\{0\}) \cap R(\{1\}) = \mathbb{N} \times \mathbb{N}$.

A *valuation on a frame* (W, R) is a $V : \mathcal{L} \longrightarrow \wp(W)$ such that for all $\varphi, \psi \in \mathcal{L}$,

- $V(\bot) = \emptyset$,
- $V(\neg\varphi) = W \setminus V(\varphi)$,
- $V(\varphi \vee \psi) = V(\varphi) \cup V(\psi)$,
- $V(\langle\varphi\rangle\psi) = \{s \in W : \exists t \in W\ (sR(V(\varphi))t\ \&\ t \in V(\psi))\}$.

A *model* is a triple consisting of a frame and a valuation on that frame. A model is *conjunctive (resp., preconjunctive, paraconjunctive)* if it is based on a conjunctive (resp., preconjunctive, paraconjunctive) frame. A *model of indiscernibility* is a model based on a frame of indiscernibility.

Example 3. The frame (W, R) where

- $W = \mathbb{R}^2$,
- for all $A \in \wp(\mathbb{R}^2)$, $R(A)$ is the binary relation on $\mathbb{R}^2$ such that for all $s, t \in \mathbb{R}^2$, $sR(A)t$ if and only if for all $u \in A$, $d(s, t) \leq d(s, u)$ where $d : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^+$ is the distance function in $\mathbb{R}^2$,

is conjunctive. For all valuations V on (W, R) and for all $\varphi, \psi \in \mathcal{L}$, if $V(\varphi) \neq \emptyset$ then $V(\langle\varphi\rangle\psi)$ is the set of all s in $\mathbb{R}^2$ such that for some t in $\mathbb{R}^2$, t is in $V(\psi)$ and the open disc with center s and radius $d(s, t)$ does not intersect $V(\varphi)$.

Example 4. The frame (W, R) where

- $W = \mathbb{R}^3$,
- for all $A \in \wp(\mathbb{R}^3)$, $R(A)$ is the binary relation on $\mathbb{R}^3$ such that for all $s, t \in \mathbb{R}^3$, $sR(A)t$ if and only if for all $u \in A$, not $L(s, t, u)$ where $L \subseteq \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$ is the collinearity relation in $\mathbb{R}^3$,

is conjunctive. For all valuations V on (W, R) and for all $\varphi, \psi \in \mathcal{L}$, if $V(\varphi) \neq \emptyset$ then $V(\langle\varphi\rangle\psi)$ is the set of all s in $\mathbb{R}^3$ such that for some t in $\mathbb{R}^3$, t is in $V(\psi)$ and the line passing through s and t does not intersect $V(\varphi)$.

The *satisfiability problem on a class $\mathcal{C}$ of frames* is the following decision problem:

input: a formula φ,
output: determine whether there exists a model (W, R, V) based on a frame in $\mathcal{C}$ such that $V(\varphi) \neq \emptyset$.

A formula φ is *true in a model* (W, R, V) (in symbols $(W, R, V) \models \varphi$) if $V(\varphi) = W$. A formula φ is *valid on a frame* (W, R) (in symbols $(W, R) \models \varphi$) if for all (W, R)-valuations V, $(W, R, V) \models \varphi$. A formula φ is *valid on a class $\mathcal{C}$ of frames* (in symbols $\mathcal{C} \models \varphi$) if for all frames (W, R) in $\mathcal{C}$, $(W, R) \models \varphi$.

Example 5. On the class of all paraconjunctive frames, the following formulas are valid: $[\bot]\varphi \to \varphi$ and $\langle\bot\rangle\varphi \to [\bot]\langle\bot\rangle\varphi$.

Example 6. On the class of all paraconjunctive frames, the following formulas are valid: $[\bot](\varphi \to \psi) \to ([\varphi]\chi \to [\psi]\chi)$.

Example 7. On the class of all frames of indiscernibility, the following formulas are valid: $[\varphi]\psi \to \psi$ and $\langle\varphi\rangle\psi \to [\varphi]\langle\varphi\rangle\psi$.

A *bounded morphism from a frame* (W, R) *to a frame* (W', R') is a $f : W \longrightarrow W'$ such that

Forward condition: for all $s, t \in W$ and for all $A \in \wp(W)$, if $sR(A)t$ then $f(s)$ $R'(f[A])f(t)$,
Backward condition: for all $s \in W$, for all $t' \in W'$ and for all $A \in \wp(W)$, if $f(s)R'(f[A])t'$ then there exists $t \in W$ such that $sR(A)t$ and $f(t) = t'$.

Lemma 4. *For all frames (W, R) and (W', R') and for all bounded morphisms f from (W, R) to (W', R'), if f is surjective then for all valuations V' on (W', R'), the $V : \mathcal{L} \longrightarrow \wp(W)$ such that for all $\varphi \in \mathcal{L}$, $V(\varphi) = f^{-1}[V'(\varphi)]$ is a valuation on (W, R).*

Lemma 5. *For all frames (W, R) and (W', R') and for all bounded morphisms f from (W, R) to (W', R'), if f is surjective then for all formulas φ, if $(W, R) \models \varphi$ then $(W', R') \models \varphi$.*

4 Axiomatizations

A *unidimensional parametrized modal logic (UPML)* is a set of formulas containing the following formulas:

($\mathbf{A}_1$) all formulas obtained from propositional tautologies after having uniformly replaced their atomic formulas by arbitrary formulas,
($\mathbf{A}_2$) $[\varphi](\psi \to \chi) \to ([\varphi]\psi \to [\varphi]\chi)$,

and closed under the following rules:

($\mathbf{R}_1$) $\frac{\varphi,\ \varphi \to \psi}{\psi}$,
($\mathbf{R}_2$) $\frac{\varphi}{[\psi]\varphi}$,
($\mathbf{R}_3$) $\frac{\varphi \leftrightarrow \psi}{[\varphi]\chi \leftrightarrow [\psi]\chi}$.

A UPML is *conjunctive* if it contains the following formulas:

($\mathbf{A}_3$) $[\bot]\varphi \to \varphi$, $\langle\bot\rangle\varphi \to [\bot]\langle\bot\rangle\varphi$,
($\mathbf{A}_4$) $[\bot](\varphi \to \psi) \to ([\varphi]\chi \to [\psi]\chi)$.

Let $\mathbf{K_g}$ (resp., $\mathbf{K_c}$) be the least UPML (resp., the least conjunctive UPML). Let $\mathbf{S}5_\mathbf{g}$ (resp., $\mathbf{S}5_\mathbf{c}$) be the least UPML (resp., the least conjunctive UPML) containing the following formulas:

($\mathbf{A}_5$) $[\varphi]\psi \to \psi$, $\langle\varphi\rangle\psi \to [\varphi]\langle\varphi\rangle\psi$.

For all UPMLs $\mathbf{L}$ and for all sets Σ of formulas, let $\mathbf{L}+\Sigma$ be the least UPML containing $\mathbf{L} \cup \Sigma$. A UPML $\mathbf{L}$ is *consistent* if $\mathbf{L} \neq \mathcal{L}$. For all UPMLs $\mathbf{L}$, we will say that a set s of formulas is $\mathbf{L}$-*consistent* if for all $n \in \mathbb{N}$ and for all $\varphi_1, \ldots, \varphi_n \in s$, $\neg(\varphi_1 \wedge \ldots \wedge \varphi_n) \notin \mathbf{L}$. Notice that for all consistent UPMLs $\mathbf{L}$, $\mathbf{L}$ is a $\mathbf{L}$-consistent set of formulas.

Lemma 6. *For all UPMLs $\mathbf{L}$ and for all $\mathbf{L}$-consistent sets s of formulas, there exists a maximal $\mathbf{L}$-consistent set t of formulas such that $s \subseteq t$.*

Lemma 7. *For all UPMLs $\mathbf{L}$, for all maximal $\mathbf{L}$-consistent sets s of formulas and for all formulas φ, ψ, if $\langle\varphi\rangle\psi \in s$ then $[\varphi]s \cup \{\psi\}$ is a $\mathbf{L}$-consistent set of formulas.*

A UPML $\mathbf{L}$ is *sound with respect to a class $\mathcal{C}$ of frames* if for all formulas φ, if $\varphi \in \mathbf{L}$ then $\mathcal{C} \models \varphi$. A UPML $\mathbf{L}$ is *complete with respect to a class $\mathcal{C}$ of frames* if for all formulas φ, if $\mathcal{C} \models \varphi$ then $\varphi \in \mathbf{L}$. The proofs of the soundness statements expressed in Proposition 1 are as expected.

Proposition 1. *In Table 1, the UPMLs listed in the left column are sound with respect to the corresponding classes of frames listed in the right column.*

As for the proofs of the corresponding completeness statements, they are not so obvious, especially when the considered UPMLs are conjunctive. Indeed, the problem with conjunctive UPMLs is that the operation of intersection — which is used in conjunctive frames for the interpretation of the modalities — is not modally definable [1, 15].

UPMLs	Classes of frames
$\mathbf{K_g}$	All frames
$\mathbf{S5_g}$	All frames of indiscernibility
$\mathbf{K_c}$	All paraconjunctive frames All preconjunctive frames All conjunctive frames
$\mathbf{S5_c}$	All paraconjunctive frames of indiscernibility All preconjunctive frames of indiscernibility All conjunctive frames of indiscernibility

Table 1.

5 Completeness: the general case

From now on in this section, we will assume that $\mathbf{L}$ is a consistent UPML. Let (W_g, R_g) be the couple where

- W_g is the set of all maximal $\mathbf{L}$-consistent sets of formulas,
- $R_g : \wp(W_g) \longrightarrow \wp(W_g \times W_g)$ is such that for all $A \in \wp(W_g)$ and for all $s, t \in W_g$, $sR_g(A)t$ if and only if for all formulas φ, if $\widehat{\varphi} = A$ then $[\varphi]s \subseteq t$ where $\widehat{\varphi}$ denotes the set of all $u \in W_g$ such that $\varphi \in u$.

Lemma 8. *For all formulas φ, ψ, if $\widehat{\varphi} = \widehat{\psi}$ then $\varphi \leftrightarrow \psi \in \mathbf{L}$.*

Since $\mathbf{L}$ is a $\mathbf{L}$-consistent set of formulas, by Lemma 6, W_g is nonempty.

Lemma 9. *(W_g, R_g) is a frame.*

Lemma 10. *If $\mathbf{L}$ contains $\mathbf{S5_g}$ then (W_g, R_g) is a frame of indiscernibility.*

Let $V_g : \mathcal{L} \longrightarrow \wp(W_g)$ be such that for all formulas φ, $V_g(\varphi) = \widehat{\varphi}$.

Lemma 11 (Truth Lemma: the general case). *(W_g, R_g, V_g) is a model.*

Proposition 2 is a consequence of Lemmas 6, 9, 10 and 11.

Proposition 2.
- $\mathbf{K_g}$ *is complete with respect to the class of all frames,*
- $\mathbf{S5_g}$ *is complete with respect to the class of all frames of indiscernibility.*

6 Completeness: the conjunctive case

From now on in this section, we will assume that $\mathbf{L}$ is a consistent conjunctive UPML.

Lemma 12. *For all maximal $\mathbf{L}$-consistent sets s, t, u of formulas, $[\bot]s \subseteq s$ and if $[\bot]s \subseteq t$ and $[\bot]s \subseteq u$ then $[\bot]t \subseteq u$.*

Let s_0 be a maximal $\mathbf{L}$-consistent set of formulas. Let (W_c, R_c) be the couple where

- W_c is the set of all maximal $\mathbf{L}$-consistent sets s of formulas such that $[\bot]s_0 \subseteq s$,

- $R_c : \wp(W_c) \longrightarrow \wp(W_c \times W_c)$ is such that for all $A \in \wp(W_c)$ and for all $s, t \in W_c$, $sR_c(A)t$ if and only if for all formulas φ, if $\widehat{\varphi} \subseteq A$ then $[\varphi]s \subseteq t$ where $\widehat{\varphi}$ denotes the set of all $u \in W_c$ such that $\varphi \in u$.

Lemma 13. *For all formulas φ, ψ,*

- *if $\widehat{\varphi} \subseteq \widehat{\psi}$ then for all $s \in W_c$, $[\bot](\varphi \to \psi) \in s$,*
- *if $\widehat{\varphi} = \emptyset$ then for all $s, t \in W_c$, $[\varphi]s \subseteq t$.*

Since s_0 is a maximal $\mathbf{L}$-consistent set of formulas, by Lemma 12, W_c is nonempty.

Lemma 14. *(W_c, R_c) is a paraconjunctive frame.*

Lemma 15. *If $\mathbf{L}$ contains $\mathbf{S5_c}$ then (W_c, R_c) is a paraconjunctive frame of indiscernibility.*

Let $V_c : \mathcal{L} \longrightarrow \wp(W_c)$ be such that for all formulas φ, $V_c(\varphi) = \widehat{\varphi}$.

Lemma 16 (Truth Lemma: the paraconjunctive case). *(W_c, R_c, V_c) is a model.*

Proposition 3 is a consequence of Lemmas 6, 14, 15 and 16.

Proposition 3.
- $\mathbf{K_c}$ *is complete with respect to the class of all paraconjunctive frames,*
- $\mathbf{S5_c}$ *is complete with respect to the class of all paraconjunctive frames of indiscernibility.*

Now, let us turn to the completeness of $\mathbf{K_c}$ with respect to the class of all preconjunctive frames and the class of all conjunctive frames and the completeness of $\mathbf{S5_c}$ with respect to the class of all preconjunctive frames of indiscernibility and the class of all conjunctive frames of indiscernibility. In this respect, Lemmas 17 and 18 will be our key results.

Lemma 17. *Let (W, R) be a paraconjunctive frame. There exist a conjunctive frame (W', R') and a surjective bounded morphism from (W', R') to (W, R).*

Proof. This proof ends after Claim 6. Let Λ be the set of all $\tau : \wp(W) \times W \longrightarrow \{0, 1\}$. Let (W', R') be the couple where

- $W' = W \times \Lambda$,
- $R' : \wp(W') \longrightarrow \wp(W' \times W')$ is such that for all $A' \in \wp(W')$ and for all $(s, \sigma), (t, \tau) \in W'$, $(s, \sigma)R'(A')(t, \tau)$ if and only if for all $A \in \wp(W)$,
  - if $A' \cap (A \times \Lambda) \neq \emptyset$ then $sR(A)t$ if and only if for all $u \in A$, $\sigma(A, u) = \tau(A, u)$,
  - for all $(u, \upsilon) \in A' \cap (A \times \Lambda)$, $\sigma(A, u) = \tau(A, u)$.

Claim 1. For all $A' \in \wp(W')$, $R'(A') = \bigcap\{R'(\{(u, \upsilon)\}) : (u, \upsilon) \in A'\}$.

Proof. Let $A' \in \wp(W')$. We demonstrate $R'(A') \supseteq \bigcap\{R'(\{(u, \upsilon)\}) : (u, \upsilon) \in A'\}$, the "$\subseteq$" direction being left as an exercise for the reader. Arguing by contradiction, suppose $R'(A') \not\supseteq \bigcap\{R'(\{(u, \upsilon)\}) : (u, \upsilon) \in A'\}$. Hence, there exist $(s, \sigma), (t, \tau) \in W'$ such that not $(s, \sigma)R'(A')(t, \tau)$ and for all $(u, \upsilon) \in A'$, $(s, \sigma)R'(\{(u, \upsilon)\})(t, \tau)$. Thus, for all $(u, \upsilon) \in A'$ and for all $A \in \wp(W)$,

- if $\{(u,v)\} \cap (A \times \Lambda) \neq \emptyset$ then $sR(A)t$ if and only if for all $v \in A$, $\sigma(A,v) = \tau(A,v)$,
- for all $(v,\omega) \in \{(u,v)\} \cap (A \times \Lambda)$, $\sigma(A,v) = \tau(A,v)$.

Consequently, for all $A \in \wp(W)$,

- if $A' \cap (A \times \Lambda) \neq \emptyset$ then $sR(A)t$ if and only if for all $v \in A$, $\sigma(A,v) = \tau(A,v)$,
- for all $(v,\omega) \in A' \cap (A \times \Lambda)$, $\sigma(A,v) = \tau(A,v)$.

Hence, $(s,\sigma)R'(A')(t,\tau)$: a contradiction.

Claim 2 is a consequence of Claim 1.

Claim 2. (W', R') is a conjunctive frame.

Let $f : W' \longrightarrow W$ be such that for all $(s,\sigma) \in W'$, $f(s,\sigma) = s$.

Claim 3. $f : W' \longrightarrow W$ is surjective.

Notice that for all $A \in \wp(W)$, $f^{-1}[A] = A \times \Lambda$.

Claim 4. For all $(s,\sigma), (t,\tau) \in W'$ and for all $A' \in \wp(W')$, if $(s,\sigma)R'(A')(t,\tau)$ then $sR(f[A'])t$.

Proof. Let $(s,\sigma), (t,\tau) \in W'$ and $A' \in \wp(W')$. Suppose $(s,\sigma)R'(A')(t,\tau)$. Arguing by contradiction, suppose not $sR(f[A'])t$. Hence, $f[A'] \neq \emptyset$. Thus, $A' \cap (f[A'] \times \Lambda) \neq \emptyset$. Since $(s,\sigma)R'(A')(t,\tau)$, $sR(f[A'])t$ if and only if for all $v \in f[A']$, $\sigma(f[A'],v) = \tau(f[A'],v)$. Moreover, for all $(u,v) \in A' \cap (f[A'] \times \Lambda)$, $\sigma(f[A'],u) = \tau(f[A'],u)$. Consequently, for all $u \in f[A']$, $\sigma(f[A'],u) = \tau(f[A'],u)$. Since $sR(f[A'])t$ if and only if for all $v \in f[A']$, $\sigma(f[A'],v) = \tau(f[A'],v)$, $sR(f[A'])t$: a contradiction.

Claim 5. For all $(s,\sigma) \in W'$, for all $t \in W$ and for all $A' \in \wp(W')$, if $sR(f[A'])t$ then there exists $\tau \in \Lambda$ such that $(s,\sigma)R'(A')(t,\tau)$.

Proof. Let $(s,\sigma) \in W'$, $t \in W$ and $A' \in \wp(W')$. Suppose $sR(f[A'])t$. We demonstrate there exists $\tau \in \Lambda$ such that $(s,\sigma)R'(A')(t,\tau)$. Indeed, we are looking for a $\tau : \wp(W) \times W \longrightarrow \{0,1\}$ such that for all $B \in \wp(W)$,

($\mathbf{C_1}$) if $A' \cap (B \times \Lambda) \neq \emptyset$ then $sR(B)t$ if and only if for all $v \in B$, $\sigma(B,v) = \tau(B,v)$,
($\mathbf{C_2}$) for all $(v,\omega) \in A' \cap (B \times \Lambda)$, $\sigma(B,v) = \tau(B,v)$.

For all $B \in \wp(W)$, let $\tau^B : W \longrightarrow \{0,1\}$ be defined as follows:

Case "$sR(B)t$": for all $v \in W$, let $\tau^B(v) = \sigma(B,v)$,

Case "not $sR(B)t$": let $v^B \in W$ be such that $v^B \in B$ and $v^B \notin f[A']$ (such v^B exists for otherwise $B \subseteq f[A']$ and not $sR(f[A'])t$) and for all $v \in W$,

- if $v \neq v^B$ then let $\tau^B(v) = \sigma(B,v)$,
- otherwise, let $\tau^B(v) = 1 - \sigma(B,v)$.

Let $\tau :\ \wp(W) \times W \longrightarrow \{0,1\}$ be such that for all $B \in \wp(W)$ and for all $v \in W$, $\tau(B,v) = \tau^B(v)$. Now, we just have to verify that for all $B \in \wp(W)$, $(\mathbf{C_1})$ and $(\mathbf{C_2})$ hold. Let $B \in \wp(W)$. About $(\mathbf{C_1})$, suppose $A' \cap (B \times \Lambda) \neq \emptyset$ and consider the following two cases: "$sR(B)t$" and "not $sR(B)t$". In the former case, for all $v \in W$, $\tau^B(v) = \sigma(B,v)$. Hence, for all $v \in W$, $\sigma(B,v) = \tau(B,v)$. Since $sR(B)t$, $(\mathbf{C_1})$ holds. In the latter case, $\tau^B(v) = \sigma(B,v)$ for every $v \in W$ except when $v = v^B$. Thus, $\sigma(B,v) = \tau(B,v)$ for every $v \in W$ except when $v = v^B$. Since not $sR(B)t$, $(\mathbf{C_1})$ holds. As for $(\mathbf{C_2})$, it holds, seeing that for all $v \in W$, if $v \in B$ and $v \in f[A']$ then $\tau^B(v) = \sigma(B,v)$.

Claim 6 is a consequence of Claims 4 and 5.

Claim 6. $f :\ W' \longrightarrow W$ is a bounded morphism from (W',R') to (W,R).

Lemma 18. *Let (W,R) be a paraconjunctive frame of indiscernibility. There exist a conjunctive frame of indiscernibility (W',R') and a surjective bounded morphism from (W',R') to (W,R).*

Proof. This proof ends after Claim 12. Let $\det :\ \wp(W) \times W \times W \longrightarrow \wp(W)$ be such that for all $A \in \wp(W)$ and for all $s,t \in W$, $\det(A,s,t) = [s]_{R(A)} \oplus [t]_{R(A)}$ where $[s]_{R(A)}$ and $[t]_{R(A)}$ are the equivalence classes of s and t modulo $R(A)$ and $\oplus$ is the operation of symmetric difference in $\wp(W)$. Notice that for all $A \in \wp(W)$ and for all $s,t \in W$, $\det(A,s,t) = \emptyset$ if and only if $sR(A)t$. Let Λ be the set of all $\tau :\ \wp(W) \times W \longrightarrow \wp(W)$ such that for all $A \in \wp(W)$, $\{s \in W :\ \tau(A,s) \neq \emptyset\}$ is finite. Let (W',R') be the couple where

- $W' = W \times \Lambda$,
- $R' :\ \wp(W') \longrightarrow \wp(W' \times W')$ is such that for all $A' \in \wp(W')$ and for all $(s,\sigma),(t,\tau) \in W'$, $(s,\sigma)R'(A')(t,\tau)$ if and only if for all $A \in \wp(W)$,
  - if $A' \cap (A \times \Lambda) \neq \emptyset$ then $\bigoplus\{\sigma(A,u) \oplus \tau(A,u) :\ u \in A\} = \det(A,s,t)$ where $\bigoplus\{\sigma(A,u) \oplus \tau(A,u) :\ u \in A\}$ denotes $\sigma(A,u_1) \oplus \tau(A,u_1) \oplus \ldots \oplus \sigma(A,u_N) \oplus \tau(A,u_N)$, $(u_1,\ldots,u_N)$ being the list of all $u \in A$ such that $\sigma(A,u) \neq \tau(A,u)$,
  - for all $(u,v) \in A' \cap (A \times \Lambda)$, $\sigma(A,u) \oplus \tau(A,u) = \emptyset$.

Claim 7. For all $A' \in \wp(W')$, $R'(A') = \bigcap\{R'(\{(u,v)\}) :\ (u,v) \in A'\}$.

Proof. Let $A' \in \wp(W')$. We demonstrate $R'(A') \supseteq \bigcap\{R'(\{(u,v)\}) :\ (u,v) \in A'\}$, the "$\subseteq$" direction being left as an exercise for the reader. Arguing by contradiction, suppose $R'(A') \not\supseteq \bigcap\{R'(\{(u,v)\}) :\ (u,v) \in A'\}$. Hence, there exist $(s,\sigma),(t,\tau) \in W'$ such that not $(s,\sigma)R'(A')(t,\tau)$ and for all $(u,v) \in A'$, $(s,\sigma)R'(\{(u,v)\})(t,\tau)$. Thus, for all $(u,v) \in A'$ and for all $A \in \wp(W)$,

- if $\{(u,v)\} \cap (A \times \Lambda) \neq \emptyset$ then $\bigoplus\{\sigma(A,v) \oplus \tau(A,v) :\ v \in A\} = \det(A,s,t)$,
- for all $(v,\omega) \in \{(u,v)\} \cap (A \times \Lambda)$, $\sigma(A,v) \oplus \tau(A,v) = \emptyset$.

Consequently, for all $A \in \wp(W)$,

- if $A' \cap (A \times \Lambda) \neq \emptyset$ then $\bigoplus\{\sigma(A,v) \oplus \tau(A,v) :\ v \in A\} = \det(A,s,t)$,

– for all $(v,\omega) \in A' \cap (A \times \Lambda)$, $\sigma(A,v) \oplus \tau(A,v) = \emptyset$.

Hence, $(s,\sigma)R'(A')(t,\tau)$: a contradiction.

Claim 8 is a consequence of Claim 7 and of the fact that for all $A \in \wp(W)$ and for all $s,t,u \in W$, $\det(A,s,s) = \emptyset$ and $\det(A,s,t) \oplus \det(A,s,u) = \det(A,t,u)$.

Claim 8. (W',R') is a conjunctive frame of indiscernibility.

Let $f:\ W' \longrightarrow W$ be such that for all $(s,\sigma) \in W'$, $f(s,\sigma) = s$.

Claim 9. $f:\ W' \longrightarrow W$ is surjective.

Notice that for all $A \in \wp(W)$, $f^{-1}[A] = A \times \Lambda$.

Claim 10. For all $(s,\sigma),(t,\tau) \in W'$ and for all $A' \in \wp(W')$, if $(s,\sigma)R'(A')(t,\tau)$ then $sR(f[A'])t$.

Proof. Let $(s,\sigma),(t,\tau) \in W'$ and $A' \in \wp(W')$. Suppose $(s,\sigma)R'(A')(t,\tau)$. Arguing by contradiction, suppose not $sR(f[A'])t$. Hence, $f[A'] \neq \emptyset$. Thus, $A' \cap (f[A'] \times \Lambda) \neq \emptyset$. Since $(s,\sigma)R'(A')(t,\tau)$, $\bigoplus\{\sigma(f[A'],u) \oplus \tau(f[A'],u)\ :\ u \in f[A']\} = \det(f[A'],s,t)$. Moreover, for all $(u,v) \in A' \cap (f[A'] \times \Lambda)$, $\sigma(f[A'],u) \oplus \tau(f[A'],u) = \emptyset$. Consequently, for all $u \in f[A']$, $\sigma(f[A'],u) \oplus \tau(f[A'],u) = \emptyset$. Hence, $\bigoplus\{\sigma(f[A'],u) \oplus \tau(f[A'],u)\ :\ u \in f[A']\} = \emptyset$. Since $\bigoplus\{\sigma(f[A'],u) \oplus \tau(f[A'],u)\ :\ u \in f[A']\} = \det(f[A'],s,t)$, $\det(f[A'],s,t) = \emptyset$. Thus, $sR(f[A'])t$: a contradiction.

Claim 11. For all $(s,\sigma) \in W'$, for all $t \in W$ and for all $A' \in \wp(W')$, if $sR(f[A'])t$ then there exists $\tau \in \Lambda$ such that $(s,\sigma)R'(A')(t,\tau)$.

Proof. Let $(s,\sigma) \in W'$, $t \in W$ and $A' \in \wp(W')$. Suppose $sR(f[A'])t$. We demonstrate there exists $\tau \in \Lambda$ such that $(s,\sigma)R'(A')(t,\tau)$. Indeed, we are looking for a $\tau:\ \wp(W) \times W \longrightarrow \wp(W)$ such that for all $B \in \wp(W)$,

($\mathbf{C_0}$) $\{u \in W\ :\ \tau(B,u) \neq \emptyset\}$ is finite,
($\mathbf{C_1}$) if $A' \cap (B \times \Lambda) \neq \emptyset$ then $\bigoplus\{\sigma(B,u) \oplus \tau(B,u)\ :\ u \in B\} = \det(B,s,t)$,
($\mathbf{C_2}$) for all $(v,\omega) \in A' \cap (B \times \Lambda)$, $\sigma(B,v) \oplus \tau(B,v) = \emptyset$.

For all $B \in \wp(W)$, let $\tau^B:\ W \longrightarrow \wp(W)$ be defined as follows:

Case "$B \subseteq f[A']$**":** for all $v \in W$, let $\tau^B(v) = \sigma(B,v)$,

Case "$B \not\subseteq f[A']$**":** let $v^B \in W$ be such that $v^B \in B$ and $v^B \notin f[A']$ and for all $v \in W$,

– if $v \neq v^B$ then let $\tau^B(v) = \sigma(B,v)$,
– otherwise, let $\tau^B(v) = \sigma(B,v) \oplus \det(B,s,t)$.

Let $\tau\ :\ \wp(W) \times W \longrightarrow \wp(W)$ be such that for all $B \in \wp(W)$ and for all $v \in W$, $\tau(B,v) = \tau^B(v)$. Now, we just have to verify that for all $B \in \wp(W)$, ($\mathbf{C_0}$), ($\mathbf{C_1}$) and ($\mathbf{C_2}$) hold. Let $B \in \wp(W)$. Concerning ($\mathbf{C_0}$), it holds, seeing that $\tau^B(v) = \sigma(B,v)$ for every $v \in W$ except when $B \not\subseteq f[A']$ and $v = v^B$. About ($\mathbf{C_1}$), suppose

$A' \cap (B \times A) \neq \emptyset$ and consider the following two cases: "$B \subseteq f[A']$" and "$B \not\subseteq f[A']$". In the former case, since $sR(f[A'])t$, $sR(B)t$. Hence, $\det(B, s, t) = \emptyset$. Since $B \subseteq f[A']$, for all $v \in W$, $\tau^B(v) = \sigma(B, v)$. Thus, for all $w \in W$, $\sigma(B, v) \oplus \tau(B, v) = \emptyset$. Consequently, $\bigoplus\{\sigma(B, v) \oplus \tau(B, v) : \ v \in B\} = \emptyset$. Since $\det(B, s, t) = \emptyset$, $(\mathbf{C_1})$ holds. In the latter case, $\tau^B(v) = \sigma(B, v)$ for every $v \in W$ except when $v = v^B$. Hence, $\bigoplus\{\sigma(B, v) \oplus \tau(B, v) : \ v \in B\} = \sigma(B, v^B) \oplus \tau(B, v^B)$. Since $\tau^B(v^B) = \sigma(B, v^B) \oplus \det(B, s, t)$, $(\mathbf{C_1})$ holds. As for $(\mathbf{C_2})$, it holds, seeing that for all $v \in W$, if $v \in B$ and $v \in f[A']$ then $\tau^B(v) = \sigma(B, v)$.

Claim 12 is a consequence of Claims 10 and 11.

Claim 12. $f : \ W' \longrightarrow W$ is a bounded morphism from (W', R') to (W, R).

Proposition 4 is a consequence of Lemmas 5, 17 and 18 and Proposition 3.

Proposition 4.
- $\mathbf{K_c}$ *is complete with respect to the class of all preconjunctive frames and the class of all conjunctive frames,*
- $\mathbf{S5_c}$ *is complete with respect to the class of all preconjunctive frames of indiscernibility and the class of all conjunctive frames of indiscernibility.*

7 Filtrations

The *equivalence setting determined by a model* (W, R, V) *and a closed set* Σ *of formulas* is the equivalence relation $\bowtie$ on W defined by

- $s \bowtie t$ if and only if for all formulas φ in Σ, $s \in V(\varphi)$ if and only if $t \in V(\varphi)$.

For all models (W, R, V), for all closed sets Σ of formulas and for all $s \in W$, the equivalence class of s modulo $\bowtie$ will be denoted $[s]$. For all models (W, R, V), for all closed sets Σ of formulas and for all $A \in \wp(W)$, the quotient of A modulo $\bowtie$ will be denoted $A/\bowtie$. A model (W', R', V') is a *filtration of a model* (W, R, V) *with respect to a closed set* Σ *of formulas* if

- $W' = W/\bowtie$,
- for all formulas φ, ψ, if $\langle\varphi\rangle\psi \in \Sigma$ then for all $s, t \in W$,
  - if $sR(V(\varphi))t$ then $[s]R'(V(\varphi)/\bowtie)[t]$,
  - if $[s]R'(V(\varphi)/\bowtie)[t]$ and $t \in V(\psi)$ then $s \in V(\langle\varphi\rangle\psi)$,
- for all atomic formulas p, if $p \in \Sigma$ then $V'(p) = V(p)/\bowtie$.

Lemma 19. *If the model* (W', R', V') *is a filtration of the model* (W, R, V) *with respect to a closed set* Σ *of formulas then for all formulas* φ*, if* $\varphi \in \Sigma$ *then* $V'(\varphi) = V(\varphi)/\bowtie$*.*

Now, let us turn to the decidability of the satisfiability problem on the class of all frames, the class of all frames of indiscernibility, the class of all conjunctive frames and the class of all conjunctive frames of indiscernibility. In this respect, Lemmas 20–23 will be our key results.

Lemma 20. *Let Σ be a closed set of formulas and (W, R, V) be a model. There exists a model (W', R', V') such that (W', R', V') is a filtration of (W, R, V) with respect to Σ.*

Proof. This proof ends after Claim 14. Let (W', R', V') be a model such that

- $W' = W/\bowtie$,
- $R' : \wp(W') \longrightarrow \wp(W' \times W')$ is such that for all $A' \in \wp(W')$ and for all $s, t \in W$, $[s]R'(A')[t]$ if and only if for all formulas φ, ψ, if $\langle\varphi\rangle\psi \in \Sigma$ and $V(\varphi)/\bowtie = A'$ then there exist $u, v \in W$ such that $s \bowtie u$, $t \bowtie v$ and $uR(V(\varphi))v$,
- for all atomic formulas p, if $p \in \Sigma$ then $V'(p) = V(p)/\bowtie$.

Claim 13. For all formulas φ, ψ, if $\langle\varphi\rangle\psi \in \Sigma$ then for all $s, t \in W$, if $sR(V(\varphi))t$ then $[s]R'(V(\varphi)/\bowtie)[t]$.

Proof. Let φ, ψ be formulas. Suppose $\langle\varphi\rangle\psi \in \Sigma$. We demonstrate for all $s, t \in W$, if $sR(V(\varphi))t$ then $[s]R'(V(\varphi)/\bowtie)[t]$. Let $s, t \in W$. Suppose $sR(V(\varphi))t$. We demonstrate $[s]R'(V(\varphi)/\bowtie)[t]$. Arguing by contradiction, suppose not $[s]R'(V(\varphi)/\bowtie)[t]$. Hence, there exist formulas φ', ψ' such that $\langle\varphi'\rangle\psi' \in \Sigma$, $V(\varphi')/\bowtie = V(\varphi)/\bowtie$ and for all $u, v \in W$, if $s \bowtie u$ and $t \bowtie v$ then not $uR(V(\varphi'))v$. Thus, $V(\varphi') = V(\varphi)$. Moreover, not $sR(V(\varphi'))t$. Consequently, not $sR(V(\varphi))t$: a contradiction.

Claim 14. For all formulas φ, ψ, if $\langle\varphi\rangle\psi \in \Sigma$ then for all $s, t \in W$, if $[s]R'(V(\varphi)/\bowtie)$ $[t]$ and $t \in V(\psi)$ then $s \in V(\langle\varphi\rangle\psi)$.

Proof. Let φ, ψ be formulas. Suppose $\langle\varphi\rangle\psi \in \Sigma$. We demonstrate for all $s, t \in W$, if $[s]R'(V(\varphi)/\bowtie)[t]$ and $t \in V(\psi)$ then $s \in V(\langle\varphi\rangle\psi)$. Let $s, t \in W$. Suppose $[s]R'(V(\varphi)/\bowtie)[t]$ and $t \in V(\psi)$. We demonstrate $s \in V(\langle\varphi\rangle\psi)$. Since $[s]R'(V(\varphi)/\bowtie)$ $[t]$, there exist $u, v \in W$ such that $s \bowtie u$, $t \bowtie v$ and $uR(V(\varphi))v$. Since $t \in V(\psi)$, $v \in V(\psi)$. Since $uR(V(\varphi))v$, $u \in V(\langle\varphi\rangle\psi)$. Since $s \bowtie u$, $s \in V(\langle\varphi\rangle\psi)$.

Lemma 21. *Let Σ be a closed set of formulas and (W, R, V) be a model of indiscernibility. There exists a model (W', R', V') of indiscernibility such that (W', R', V') is a filtration of (W, R, V) with respect to Σ.*

Proof. This proof ends after Claim 16. Let (W', R', V') be a model of indiscernibility such that

- $W' = W/\bowtie$,
- $R' : \wp(W') \longrightarrow \wp(W' \times W')$ is such that for all $A' \in \wp(W')$ and for all $s, t \in W$, $[s]R'(A')[t]$ if and only if for all formulas φ, ψ, if $\langle\varphi\rangle\psi \in \Sigma$ and $V(\varphi)/\bowtie = A'$ then $s \in V(\langle\varphi\rangle\psi)$ if and only if $t \in V(\langle\varphi\rangle\psi)$,
- for all atomic formulas p, if $p \in \Sigma$ then $V'(p) = V(p)/\bowtie$.

Claim 15. For all formulas φ, ψ, if $\langle\varphi\rangle\psi \in \Sigma$ then for all $s, t \in W$, if $sR(V(\varphi))t$ then $[s]R'(V(\varphi)/\bowtie)[t]$.

Proof. Let φ, ψ be formulas. Suppose $\langle\varphi\rangle\psi \in \Sigma$. We demonstrate for all $s, t \in W$, if $sR(V(\varphi))t$ then $[s]R'(V(\varphi)/\bowtie)[t]$. Let $s, t \in W$. Suppose $sR(V(\varphi))t$. We demonstrate $[s]R'(V(\varphi)/\bowtie)[t]$. Arguing by contradiction, suppose not $[s]R'(V(\varphi)/\bowtie)[t]$. Hence, there exist formulas φ', ψ' such that $\langle\varphi'\rangle\psi' \in \Sigma$, $V(\varphi')/\bowtie = V(\varphi)/\bowtie$ and either $s \in V(\langle\varphi'\rangle\psi')$ and $t \notin V(\langle\varphi'\rangle\psi')$, or $s \notin V(\langle\varphi'\rangle\psi')$ and $t \in V(\langle\varphi'\rangle\psi')$. Without loss of generality, suppose $s \in V(\langle\varphi'\rangle\psi')$ and $t \notin V(\langle\varphi'\rangle\psi')$. Thus, there exists $u \in W$ such that $sR(V(\varphi'))u$ and $u \in V(\psi')$. Since $V(\varphi')/\bowtie = V(\varphi)/\bowtie$, $V(\varphi') = V(\varphi)$. Since $t \notin V(\langle\varphi'\rangle\psi')$ and $u \in V(\psi')$, not $tR(V(\varphi'))u$. Since $sR(V(\varphi'))u$, not $sR(V(\varphi'))t$. Since $V(\varphi') = V(\varphi)$, not $sR(V(\varphi))t$: a contradiction.

Claim 16. For all formulas φ, ψ, if $\langle\varphi\rangle\psi \in \Sigma$ then for all $s, t \in W$, if $[s]R'(V(\varphi)/\bowtie)$ $[t]$ and $t \in V(\psi)$ then $s \in V(\langle\varphi\rangle\psi)$.

Proof. Let φ, ψ be formulas. Suppose $\langle\varphi\rangle\psi \in \Sigma$. We demonstrate for all $s, t \in W$, if $[s]R'(V(\varphi)/\bowtie)[t]$ and $t \in V(\psi)$ then $s \in V(\langle\varphi\rangle\psi)$. Let $s, t \in W$. Suppose $[s]R'(V(\varphi)/\bowtie)[t]$ and $t \in V(\psi)$. We demonstrate $s \in V(\langle\varphi\rangle\psi)$. Since $[s]R'(V(\varphi)/\bowtie)$ $[t]$, $s \in V(\langle\varphi\rangle\psi)$ if and only if $t \in V(\langle\varphi\rangle\psi)$. Since $t \in V(\psi)$, $t \in V(\langle\varphi\rangle\psi)$. Since $s \in V(\langle\varphi\rangle\psi)$ if and only if $t \in V(\langle\varphi\rangle\psi)$, $s \in V(\langle\varphi\rangle\psi)$.

Lemma 22. *Let Σ be a closed set of formulas and (W, R, V) be a paraconjunctive model. There exists a paraconjunctive model (W', R', V') such that (W', R', V') is a filtration of (W, R, V) with respect to Σ.*

Proof. Let (W', R', V') be a model such that

- $W' = W/\bowtie$,
- $R' : \wp(W') \longrightarrow \wp(W' \times W')$ is such that for all $A' \in \wp(W')$ and for all $s, t \in W$, $[s]R'(A')[t]$ if and only if for all formulas φ, ψ, if $\langle\varphi\rangle\psi \in \Sigma$ and $V(\varphi)/\bowtie \subseteq A'$ then there exist $u, v \in W$ such that $s \bowtie u$, $t \bowtie v$ and $uR(V(\varphi))v$,
- for all atomic formulas p, if $p \in \Sigma$ then $V'(p) = V(p)/\bowtie$.

Now, the rest of the proof is similar to the corresponding rest of the proof of Lemma 20, the main difference being that one has to verify here that (W', R', V') is a paraconjunctive model, an exercise that we leave for the reader.

Lemma 23. *Let Σ be a closed set of formulas and (W, R, V) be a paraconjunctive model of indiscernibility. There exists a paraconjunctive model (W', R', V') of indiscernibility such that (W', R', V') is a filtration of (W, R, V) with respect to Σ.*

Proof. Let (W', R', V') be a model of indiscernibility such that

- $W' = W/\bowtie$,
- $R' : \wp(W') \longrightarrow \wp(W' \times W')$ is such that for all $A' \in \wp(W')$ and for all $s, t \in W$, $[s]R'(A')[t]$ if and only if for all formulas φ, ψ, if $\langle\varphi\rangle\psi \in \Sigma$ and $V(\varphi)/\bowtie \subseteq A'$ then $s \in V(\langle\varphi\rangle\psi)$ if and only if $t \in V(\langle\varphi\rangle\psi)$,
- for all atomic formulas p, if $p \in \Sigma$ then $V'(p) = V(p)/\bowtie$.

Now, the rest of the proof is similar to the corresponding rest of the proof of Lemma 21, the main difference being that one has to verify here that (W', R', V') is a paraconjunctive model, an exercise that we leave for the reader.

Proposition 5 is a consequence of [3, Theorem 6.7] and Lemmas 5, 17, 18 and 20–23.

Proposition 5. *The satisfiability problem is decidable on the following classes of frames:*

- *the class of all frames,*
- *the class of all frames of indiscernibility,*
- *the class of all conjunctive frames,*
- *the class of all conjunctive frames of indiscernibility.*

Conjecture 1. We believe the satisfiability problem is $\mathbf{PSPACE}$-complete on the class of all frames and the class of all frames of indiscernibility. We believe as well the satisfiability problem is $\mathbf{EXPTIME}$-complete on the class of all conjunctive frames and the class of all conjunctive frames of indiscernibility.

8 Conclusion

What has been done in this paper? Firstly, we have introduced the syntax and the semantics of a new family of modal logics: UPMLs (Sections 2 and 3). Secondly, we have axiomatically introduced different UPMLs (Section 4). Thirdly, we have proved their completeness with respect to appropriate classes of relational structures (Sections 5 and 6). In this respect, we have seen that the operation of intersection — which is used in conjunctive frames for the interpretation of the modalities — being not modally definable, our proofs of completeness are not so obvious when the considered UPMLs are conjunctive. Fourthly, we have shown the decidability of some related satisfiability problems (Section 7).

Much remains to be done. For instance,

- to import first-order ideas into conjunctive UPMLs (constructs of hybrid logics [2, 7], the difference operator [3, Section 7.1], etc),
- to develop the model theory of conjunctive UPMLs (classical definition of bisimulations [3, Section 2.2], classical definition of saturated models [3, Section 2.6], etc),
- to elaborate the correspondence theory of conjunctive UPMLs (analogue of Sahlqvist Correspondence Theorem [3, Section 3.6], analogues of Chagrova's Theorems [4, 6], etc),
- to investigate the computability of the satisfiability problem in such-and-such class of conjunctive frames and develop automatic procedures for solving it (filtration method [5, Chapter 5], tableaux-based approach [12], etc),
- to compare conjunctive UPMLs with other forms of modal logics based on parametrized connectives (knowledge representation logics [8, 14, 18], Boolean modal logic [10, 11], etc),
- to construct the duality theory of conjunctive UPMLs (standard definition of Boolean algebras with operators [13, Section 2.2], standard definition of general frames [13, Section 4.6], etc).

Other avenues of research might consist in considering that frames are couples of the form (X, τ) where X is a nonempty set and $\tau : \wp(X) \longrightarrow \wp(\wp(X))$ is such that for all $A \in \wp(X)$, $\tau(A)$ is a topology on X. In that case, a *valuation on a frame* (X, τ) will be a $V : \mathcal{L} \longrightarrow \wp(X)$ such that $V(\langle\varphi\rangle\psi) = \{s \in X : \forall \mathcal{O} \in \tau(V(\varphi))\ (s \in \mathcal{O} \Rightarrow \mathcal{O} \cap V(\psi) \neq \emptyset)\}$ among other conditions. Further investigations are needed for obtaining the UPML that will completely axiomatize the validities thus defined.

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References

1. Ågotnes, T., Wáng, Y.: *Resolving distributed knowledge.* Artificial Intelligence **252** (2017) 1–21.
2. Areces, C., ten Cate, B.: *Hybrid logics.* In *Handbook of Modal Logic.* Elsevier (2007) 821–868.
3. Blackburn, P., de Rijke, M., Venema, Y.: *Modal Logic.* Cambridge University Press (2001).
4. Chagrov, A., Chagrova, L.: *The truth about algorithmic problems in correspondence theory.* In *Advances in Modal Logic. Vol. 6.* College Publications (2006) 121–138.
5. Chagrov, A., Zakharyaschev, M.: *Modal Logic.* Oxford University Press (1997).
6. Chagrova, L.: *An undecidable problem in correspondence theory.* The Journal of Symbolic Logic **56** (1991) 1261–1272.
7. Costa, D., Martins, M.: *A four-valued hybrid logic with non-dual modal operators.* In *Dynamic Logic.* Springer (2020) 88-103.
8. Demri, S.: *A logic with relative knowledge operators.* Journal of Logic, Language and Information **8** (1999) 167–185.
9. Demri, S., Orłowska, E.: *Incomplete Information: Structure, Inference, Complexity.* Springer (2002).
10. Gargov, G., Passy, S.: *A note on Boolean modal logic.* In *Mathematical Logic.* Plenum Press (1990) 299–309.
11. Gargov, G., Passy, S., Tinchev, T.: *Modal environment for Boolean speculations.* In *Mathematical Logic and its Applications.* Plenum Press (1987) 253–263.
12. Goré, R.: *Tableau methods for modal and temporal logics.* In *Handbook of Tableau Methods.* Springer (1999) 297–396.
13. Kracht, M.: *Tools and Techniques in Modal Logic.* Elsevier (1999).
14. Orłowska, E.: *Kripke semantics for knowledge representation logics.* Studia Logica **49** (1990) 255–272.
15. Passy, S., Tinchev, T.: *An essay in combinatory dynamic logic.* Information and Computation **93** (1991) 263–332.
16. Reynolds, M.: *The complexity of temporal logic over the reals.* Annals of Pure and Applied Logic **161** (2010) 1063–1096.
17. Sistla, A., Clarke, E.: *The complexity of propositional linear temporal logics.* Journal of the Association for Computing Machinery **32** (1985) 733–749.
18. Vakarelov, D.: *Modal logics for knowledge representation systems.* Theoretical Computer Science **90** (1991) 433–456.

Appendix

This Appendix includes the proofs of some of our results. Most of these proofs are relatively simple and we have included them here just for the sake of the completeness.

Proof of Lemma 4. Similar to the proof of Bounded Morphism Lemma [3, Proposition 2.14].

Proof of Lemma 5. Consequence of Lemma 4.

Proof of Lemma 6. Similar to the proof of Lindenbaum's Lemma [5, Lemma 5.1].

Proof of Lemma 7. Similar to the proof of Existence Lemma [13, Proposition 2.8.4].

Proof of Lemma 8. Consequence of Lemma 6.

Proof of Lemma 10. Consequence of the fact that $\mathbf{S5_g}$ contains all formulas of the form $[\varphi]\psi \to \psi$ and $\langle\varphi\rangle\psi \to [\varphi]\langle\varphi\rangle\psi$.

Proof of Lemma 11. The proof that V_g satisfies the conditions for $\perp$, $\neg$ and $\vee$ is as expected. We only show that V_g satisfies the condition for $\langle\cdot\rangle$. Let φ, ψ be a formulas. Let $s \in W_g$. We only demonstrate $s \in V_g(\langle\varphi\rangle\psi)$ only if there exists $t \in W_g$ such that $sR_g(V_g(\varphi))t$ and $t \in V_g(\psi)$, the "if" direction being left as an exercise for the reader. Suppose $s \in V_g(\langle\varphi\rangle\psi)$. We demonstrate there exists $t \in W_g$ such that $sR_g(V_g(\varphi))t$ and $t \in V_g(\psi)$. Since $s \in V_g(\langle\varphi\rangle\psi)$, $\langle\varphi\rangle\psi \in s$. Let $t_0 = [\varphi]s \cup \{\psi\}$. Notice that $[\varphi]s \subseteq t_0$ and $\psi \in t_0$. By Lemma 7, t_0 is a $\mathbf{L}$-consistent set of formulas. Hence, by Lemma 6, let t be a maximal $\mathbf{L}$-consistent set of formulas such that $t_0 \subseteq t$. Since $[\varphi]s \subseteq t_0$ and $\psi \in t_0$, $[\varphi]s \subseteq t$ and $\psi \in t$. Thus, $t \in V_g(\psi)$.

Claim. $sR_g(V_g(\varphi))t$.

Proof. We demonstrate for all formulas φ', if $\widehat{\varphi'} = V_g(\varphi)$ then $[\varphi']s \subseteq t$. Let φ' be a formula. Suppose $\widehat{\varphi'} = V_g(\varphi)$. We demonstrate $[\varphi']s \subseteq t$. Let ψ' be a formula. Suppose $[\varphi']\psi' \in s$. We demonstrate $\psi' \in t$. Since $\widehat{\varphi'} = V_g(\varphi)$, by Lemma 8, $\varphi' \leftrightarrow \varphi \in \mathbf{L}$. Hence, $[\varphi']\psi' \leftrightarrow [\varphi]\psi' \in \mathbf{L}$. Since $[\varphi']\psi' \in s$, $[\varphi]\psi' \in s$. Since $[\varphi]s \subseteq t$, $\psi' \in t$.

Proof of Lemma 12. Consequence of the fact that $\mathbf{L}$ contains all formulas of the form $[\perp]\varphi \to \varphi$ and $\langle\perp\rangle\varphi \to [\perp]\langle\perp\rangle\varphi$.

Proof of Lemma 13. Let φ, ψ be formulas.

Suppose $\widehat{\varphi} \subseteq \widehat{\psi}$. We demonstrate for all $s \in W_c$, $[\perp](\varphi \to \psi) \in s$. Let $s \in W_c$. We demonstrate $[\perp](\varphi \to \psi) \in s$. Arguing by contradiction, suppose $[\perp](\varphi \to \psi) \notin s$. Hence, $\langle\perp\rangle(\varphi \wedge \neg\psi) \in s$. Let $u_0 = [\perp]s \cup \{\varphi, \neg\psi\}$. Notice that $[\perp]s \subseteq u_0$, $\varphi \in u_0$ and $\neg\psi \in u_0$. By Lemma 7, u_0 is a $\mathbf{L}$-consistent set of formulas. Thus, by Lemma 6, let u be a maximal $\mathbf{L}$-consistent set of formulas such that $u_0 \subseteq u$. Since $[\perp]s \subseteq u_0$,

$\varphi \in u_0$ and $\neg\psi \in u_0$, $[\bot]s \subseteq u$, $\varphi \in u$ and $\neg\psi \in u$. Since $[\bot]s_0 \subseteq s$, by Lemma 12, $[\bot]s_0 \subseteq u$. Consequently, $u \in W_c$. Since $\varphi \in u$ and $\neg\psi \in u$, $u \in \widehat{\varphi}$ and $\psi \notin u$. Since $\widehat{\varphi} \subseteq \widehat{\psi}$, $u \in \widehat{\psi}$. Hence, $\psi \in u$: a contradiction.

Suppose $\widehat{\varphi} = \emptyset$. We demonstrate for all $s, t \in W_c$, $[\varphi]s \subseteq t$. Let $s, t \in W_c$. We demonstrate $[\varphi]s \subseteq t$. Let χ be a formula. Suppose $[\varphi]\chi \in s$. We demonstrate $\chi \in t$. Since $\widehat{\varphi} = \emptyset$, by the previous item, $[\bot](\varphi \to \bot) \in s$. Thus, $[\varphi]\chi \to [\bot]\chi \in s$. Since $[\varphi]\chi \in s$, $[\bot]\chi \in s$. Since $[\bot]s_0 \subseteq s$, $\langle\bot\rangle[\bot]\chi \in s_0$. Consequently, $[\bot]\chi \in s_0$. Since $[\bot]s_0 \subseteq t$, $\chi \in t$.

Proof of Lemma 14. Indeed, $R_c(\emptyset) = W_c \times W_c$. Why? Simply because by Lemma 13, for all $s, t \in W_c$ and for all formulas φ, if $\widehat{\varphi} = \emptyset$ then $[\varphi]s \subseteq t$. Hence, for all $s, t \in W_c$, $sR_c(\emptyset)t$. Moreover, for all $A, B \in \wp(W_c)$, if $A \subseteq B$ then $R_c(A) \supseteq R_c(B)$. Why? Simply because for all $A, B \in \wp(W_c)$, if $A \subseteq B$ then for all formulas φ, if $\widehat{\varphi} \subseteq A$ then $\widehat{\varphi} \subseteq B$. Thus, for all $A, B \in \wp(W_c)$, if $A \subseteq B$ then for all $t, u \in W_c$, if $tR_c(B)u$ then $tR_c(A)u$.

Proof of Lemma 15. Consequence of the fact that $\mathbf{S5_c}$ contains all formulas of the form $[\varphi]\psi \to \psi$ and $\langle\varphi\rangle\psi \to [\varphi]\langle\varphi\rangle\psi$.

Proof of Lemma 16. The proof that V_c satisfies the conditions for $\bot$, $\neg$ and $\vee$ is as expected. We only show that V_c satisfies the condition for $\langle\cdot\rangle$. Let φ, ψ be formulas. Let $s \in Wc$. We only demonstrate $s \in V_c(\langle\varphi\rangle\psi)$ only if there exists $t \in W_c$ such that $sR_c(V_c(\varphi))t$ and $t \in V_c(\psi)$, the "if" direction being left as an exercise for the reader. Suppose $s \in V_c(\langle\varphi\rangle\psi)$. We demonstrate there exists $t \in W_c$ such that $sR_c(V_c(\varphi))t$ and $t \in V_c(\psi)$. Since $s \in V_c(\langle\varphi\rangle\psi)$, $\langle\varphi\rangle\psi \in s$. Let $t_0 = [\varphi]s \cup \{\psi\}$. Notice that $[\varphi]s \subseteq t_0$ and $\psi \in t_0$. By Lemma 7, t_0 is a $\mathbf{L}$-consistent set of formulas. Hence, by Lemma 6, let t be a maximal $\mathbf{L}$-consistent set of formulas such that $t_0 \subseteq t$. Since $[\varphi]s \subseteq t_0$ and $\psi \in t_0$, $[\varphi]s \subseteq t$ and $\psi \in t$. Thus, $t \in V_c(\psi)$.

Claim. $sR_c(V_c(\varphi))t$.

Proof. We demonstrate for all formulas φ', if $\widehat{\varphi'} \subseteq V_c(\varphi)$ then $[\varphi']s \subseteq t$. Let φ' be a formula. Suppose $\widehat{\varphi'} \subseteq V_c(\varphi)$. We demonstrate $[\varphi']s \subseteq t$. Let ψ' be a formula. Suppose $[\varphi']\psi' \in s$. We demonstrate $\psi' \in t$. Since $\widehat{\varphi'} \subseteq V_c(\varphi)$, by Lemma 13, $[\bot](\varphi' \to \varphi) \in s$. Hence, $[\varphi']\psi' \to [\varphi]\psi' \in s$. Since $[\varphi']\psi' \in s$, $[\varphi]\psi' \in s$. Since $[\varphi]s \subseteq t$, $\psi' \in t$.

Proof of Lemma 19. Similar to the proof of Filtration Theorem [3, Theorem 2.39].

Relating Kleene algebras with pseudo uninorms

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Abstract. This paper explores a strict relation between two core notions of the semantics of programs and of fuzzy logics: Kleene Algebras and (pseudo) uninorms. It shows that every Kleene algebra induces a pseudo uninorm, and that some pseudo uninorms induce Kleene algebras. This connection establishes a new perspective on the theory of Kleene algebras and provides a way to build (new) Kleene algebras. The latter aspect is potentially useful as a source of formalism to capture and model programs acting with fuzzy behaviours and domains.

1 Introduction

The adoption of algebraic structures and techniques to model and reason about programs has a long tradition in Computer Science, and is the basis of some of its main pillars, including Process Algebra and Abstract Data Types Specification. In particular, *Algebras of Programs*, coming from regular languages and automata theory, have been widely considered as suitable frameworks to support the rigorous semantics for analysis of algorithms and the design and development of complex systems. On the basis of this field is the notion of Kleene Algebra [14], today accepted as the standard abstraction of a computational system. Among of its examples, an algebraic framework for coherent confluence proofs, in rewriting theory, for an higher dimensional generalisation of modal Kleene algebra proposes in [3] and the algebra of the regular languages, traces of programs and the algebra of relations on which the program states transitions are modelled as binary relations on the set of states. For instance, by starting from the atomic programs represented in the transition systems of Fig. 1
we have the Kleene algebra of binary relations to interpret composed programs build from these ones. For instance, the sequential composition of these structures, i.e. the program that interpret the program $A_\pi; A_{\pi'}$ that execute one step in A_π followed by another in $A_{\pi'}$, is just interpreted as the standard relational composition, as represented in Fig 2. Moreover, operations of non deterministic choice + and iteration closure $*$, the ones needed to encode any imperative program, are also provided by the mentioned Kleene algebra.

If the mentioned above models plays a relevant role in the current formal development and design processes, the emergence of new computational paradigms

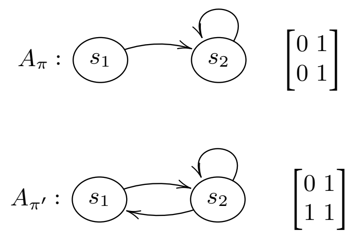

Fig. 1. Examples of abstract programs

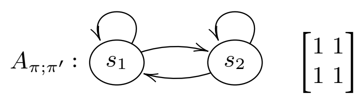

Fig. 2. Examples of abstract programs

and scenarios, as Fuzzy and Probabilistic programming, entails not only the definition of new Kleene algebra models, but also some variants and generalisations. As examples of the latter efforts, we can point out our recent development on the study of Kleene algebras to deal with "intervals as programs" [22], in order to deal with situations where the precise values of the transitions weights are not provided (e.g. entailed by the machine representation of an irrational number).

This paper develops a novel algebraic study on Kleene algebras, based on pseudo uninorms defined over partial orders. As is well known, we can easily obtain a Kleene algebra from any Boolean algebra, by taking the operation $\star$ as the as the constant function $x\star = \top$, where $\top$ is the top element of the algebra. Following this intuition, we abstract the infimum operation as pseudo uninorms defined over partial orders, in order to build algebras for fuzzy programs. As expected such new algebras generalises the classic case.

We investigate how fuzzy programs, i.e., elements of these structures, behaves with respect to the Kleene operations. At this level, classic choice is maintained, but the notions of sequential composition and Kleene closure are abstracted as specific uninorms, defined over meet semilattices.

Building new Kleene algebras from other Kleene algebras can be also very useful. The work of Conway plays a very relevant role. He introduces in [4] some matricial constructions that preserve the Kleene algebra structure. In other words, he introduces a method, with which, given a a Kleene algebra over a set K, it construct a Kleene algebra over the squared matrices $\mathrm{M}_n(K)$. For instance the Kleene algebra of relations used bellow (cf. Fig 1), which elements are the adjacency matrices, can be taken with this method from the Kleene algebra defined by the two-elements boolean algebra (with $0^\star = 1^\star = 1$).

In this work we introduce an operator to construct new Kleene algebras from other Kleene algebras based in the notion of automorphism. These maps reinterprets programs and the programs operations of a Kleene algebra into a new Kleene algebra, by contributing with an alternative source of program algebras.

Context and Contributions.

In [18] Menger introduced triangular norms (t-norms) in order to provide triangle inequality for distances on probabilistic metric spaces. Since Menger's definition is weak, Schweizer and Sklar [23] provided a new definition for t-norms adding new axioms such as associativity and taking 1 as the neutral element. In [24], they introduced the notion of t-conorm by simply taking 0 as the neutral element instead of 1.

The axioms of T-norms (T-conorms) was, then, changed by abolishing some of its conditions. Those weakening gave rise to the so called **pseudo t-norm**. In [9] (see also [7]) Siegfried Gottwald considered the notion of pseudo t-norm by abolishing the commutativity property. On the other hand, further authors such as abolished other axioms — see [11, 15, 29]. In particular, in [11] Sándor Jenei suppressed the commutativity property and the left side of the isotonicity property; and in [15], in addition to these two properties, Hua-Wen Liu suppressed the associativity. In [17], M. Mas, M. Monserrat and J. Torrens introduced the notion of left uninorms and right uninorms. One year after W. Sander [21] introduced the notion of pseudo uninorm as a bivariated function on the unit interval that is associative, isotone and has a neutral element. This notion coincides with the functions which are, both, left and right uninorms. In [28] the notion of left uninorms, right uninorms and pseudo uninorms was extended for lattice-valued sets. Two pseudo uninorms having the same neutral element is called of the **same type**. In [27] the notion of pseudo uninorm was extended for complete lattices and here we generalize them for posets. Recently, the papers [16, 25, 26] consider lattice-valued and $[0,1]$-valued pseudo uninorms.

In this paper, we investigate the notion of pseudo uninorms and show how they can be used to build Kleene Algebras, which is a kind of algebra used to model some computational systems.

Outline. This paper is organized in the following way: Section 2 introduces the notions of pseudo uninorms and Kleene algebras. Section 3 provide some new results and construction on pseudo uninorms. Section 4 shows how Kleene algebras are built from certain pseudo uninorms and that every Kleene algebra is related to a pseudo uninorm. The section also studies automorphisms on this structures and how they can generate new pseudo uninorm based Kleene algebras.

2 Preliminaries

Let $\langle P, \leq \rangle$ be a poset and $e \in P$. Then, trivially, $\langle P_e, \leq_e \rangle$ and $\langle P^e, \leq^e \rangle$ are poset with a greater and least, respectively, element when $P_e = \{x \in P : x \leq e\}$, $P^e = \{x \in P : e \leq x\}$ and $\leq_e$ and $\leq^e$ are the restriction of $\leq$ to P_e and P^e, respectively [3]. Let $\Delta_P = \{a \in P : \text{for each } x \in P\ a \leq x \text{ or } x \leq a\}$. $\langle P, \leq \rangle$ is a **total order set**, whenever $\Delta_P = P$. $\langle P, \leq \rangle$ is a **meet (join) semilattice** if every

[3] The reader can also find in the literature $\downarrow e$ and $\uparrow e$, respectively.

$x, y \in P$ have an infimum (supremum) in P, denoted by $x \wedge y$ ($x \vee y$). $\langle P, \leq \rangle$ is a lattice if it is both: meet and join semilattice.

Closure operators play an important role in several fields of the mathematics; e.g. in Algebra, Logic and Topology. In this paper a closure operator will be required to develop this work:

Definition 1. *Let $\langle P, \leq \rangle$ be a poset. A* **closure operator** *on P is a function $\star : P \to P$ such that for each $x, y \in P$*

(C1) *if $x \leq y$ then $x^\star \leq y^\star$ — Isotonicity,*
(C2) *$x \leq x^\star$ — inflation, and*
(C3) *$(x^\star)^\star = x^\star$ — idempotency.*

Definition 2. *Let $\langle P, \leq \rangle$ be a poset. A function $U : P \times P \to P$ is a* **pseudo uninorm** *on P, whenever, for each $w, x, y, z \in P$ it satisfies:*

1. *$U(x, U(y, z)) = U(U(x, y), z)$ —* ***Associativity****,*
2. *$w \leq x$ and $y \leq z$ then $U(w, y) \leq U(x, z)$ —* ***Isotonicity****, and*
3. *there is $e \in P$ s.t. $U(x, e) = U(e, x) = x$ —* ***has neutral element****.*

$\mathfrak{U}_P^e$ is the set of all pseudo uninorms on P with neutral element e. If e is the greater (least) element of P then U is called of ***pseudo t-norm*** *(****pseudo t-conorm****).*

Commutative pseudo uninorms are called of uninorm on P in [12]. Uninorms on poset $[0, 1]$ were introduced in [6], but the name uninorm only was coined in [30].

Remark 1. If $\langle P, \leq \rangle$ is a meet-semilattice, then the infimum, i.e. $\wedge$, is a pseudo t-norm iff P has a top element. Analogously, if $\langle P, \leq \rangle$ is a join-semilattice then the supremum, i.e. $\vee$, is a pseudo t-conorm iff P has a bottom element.

Remark 2. The set $\mathfrak{U}_P^e$ endowed with the following binary relation is a partial order:

$$U_1 \preceq_e U_2 \text{ iff } \forall x, y \in P,\ \ U_1(x, y) \leq U_2(x, y).$$

If $U \in \mathfrak{U}_P^e$ then $U(x, y) \leq x \leq e$ and $U(x, y) \leq y$ whenever $x, y \in P_e$, $U(x, y) \geq x \geq e$ and $U(x, y) \geq y$ whenever $x, y \in P^e$, and $x \leq U(x, y) \leq y$ (and also $x \leq U(y, x) \leq y$) whenever $x \in P_e$ and $y \in P^e$.

Remark 3. Let $\langle P, \leq \rangle$ be a bounded poset and $e \in P$. If $U \in \mathfrak{U}_P^e$ then $U(\top, \top) \geq U(\top, e) = \top$ and therefore $U(\top, \top) = \top$. Analogously it is possible to prove that $U(\bot, \bot) = \bot$.

We recall in the notion of Kleene algebra. This algebraic structure represents the abstract notion of a computational systems where programs can be modelled. Namely it is constituted by an universe of programs K that can be operated by a (non deterministic) choice $+$, by a sequential composition ; and by an iterative closure $*$. The algebra of regular languages, of binary relations and of program traces are well known instantiations of such structure.

Definition 3. *An algebra* $\langle K, +, \cdot, \star, 0, 1\rangle$ *of type* $(2, 2, 1, 0, 0)$ *is a Kleene algebra if for each* $a, b, c \in K$ *satisfy the following axioms:*

- ***(KA1)*** $a + (b + c) = (a + b) + c$;
- ***(KA2)*** $a + b = b + a$;
- ***(KA3)*** $a + a = a$;
- ***(KA4)*** $a + 0 = 0 + a = a$;
- ***(KA5)*** $a \cdot (b \cdot c) = (a \cdot b) \cdot c$;
- ***(KA6)*** $a \cdot 1 = 1 \cdot a = a$;
- ***(KA7)*** $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$;
- ***(KA8)*** $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$;
- ***(KA9)*** $a \cdot 0 = 0 \cdot a = 0$;
- ***(KA10)*** $1 + (a \cdot a^\star) \leq a^\star$;
- ***(KA11)*** $1 + (a^\star \cdot a) \leq a^\star$;
- ***(KA12)*** *If* $a \cdot b \leq b$ *then* $a^\star \cdot b \leq b$; *and*
- ***(KA13)*** *If* $a \cdot b \leq a$ *then* $a \cdot b^\star \leq a$.

Where $\leq$ *is the natural partial order on* K *defined by*

$$a \leq b \text{ if and only if } a + b = b. \tag{1}$$

Remark 4. In fact $\langle K, \leq\rangle$ is a join-semilattice with 0 as least element [14].

Lemma 1. *[14] Let* $\langle K, +, \cdot, \star, 0, 1\rangle$ *be a Kleene algebra. Then*

- ***(KO1)*** *If* $a \leq b$ *then* $a^\star \leq b^\star$.
- ***(KO2)*** $0^\star = 1$.
- ***(KO3)*** $1 + a \cdot a^\star = a^\star$.
- ***(KO4)*** $(a^\star)^\star = a^\star$.

3 Some new results and construction on pseudo uninorms

Proposition 1. *Let* $\langle P, \leq\rangle$ *be a poset with a bottom element* $\perp$. *For each* $e \in P$, *if* $\langle P^e, \leq^e\rangle$ *is a join-semilattice, then* $\langle \mathfrak{U}_P^e, \preceq_e\rangle$ *has a bottom element.*

Proof. Let $\perp$ be the least element of P. Then the function

$$\underline{U}_e(x, y) = \begin{cases} \perp & \text{if } x, y \notin P^e \\ x \vee y & \text{if } x, y \in P^e \\ x & \text{if } x \notin P^e \text{ and } y \in P^e \\ y & \text{if } x \in P^e \text{ and } y \notin P^e \end{cases}$$

is the bottom element of $\langle \mathfrak{U}_P^e, \preceq_e\rangle$.

Proposition 2. *Let* $\langle P, \leq\rangle$ *be a poset with a greater element* $\top$. *For each* $e \in P$ *if* $\langle P_e, \preceq_e\rangle$ *is a meet-semilattice then* $\langle \mathfrak{U}_P^e, \preceq_e\rangle$ *has a greater element.*

Proof. The function

$$\overline{U}_e(x,y) = \begin{cases} x \wedge y & \text{if } x,y \in P_e \\ \top & \text{if } x,y \notin P_e \\ x & \text{if } x \notin P_e \text{ and } y \in P_e \\ y & \text{if } x \in P_e \text{ and } y \notin P_e \end{cases}$$

is the greater element of $\langle \mathfrak{U}_P^e, \preceq_e \rangle$.

Proposition 3. *Let $\langle P, \leq \rangle$ be a poset, $e \in P$ and $U \in \mathfrak{U}_P^e$. Then the restriction, $U_{/P_e}$, is a pseudo t-norm on $\langle P_e, \leq_e \rangle$ and $U_{/P^e}$ is a pseudo t-conorm on $\langle P^e, \leq^e \rangle$.*

Proof. Straightforward.

Corollary 1. *Let $\langle P, \leq \rangle$ be a poset, $e \in P$, $U \in \mathfrak{U}_P^e$. Then for each isotone bijection $\phi : P_e \to P$, the function $T : P \times P \to P$ defined by*

$$T(x,y) = \phi(U(\phi^{-1}(x), \phi^{-1}(y)))$$

is a pseudo t-norm on P.

Corollary 2. *Let $\langle P, \leq \rangle$ be a poset, $e \in P$, $U \in \mathfrak{U}_P^e$. Then for each isotone bijection $\psi : P^e \to P$, the function $S : P \times P \to P$ defined by*

$$S(x,y) = \psi(U(\psi^{-1}(x), \psi^{-1}(y)))$$

is a pseudo t-conorm on P.

Proposition 4. *Let $\langle P, \leq \rangle$ be a poset, $\mathfrak{U}_P$ the set of all pseudo uninorms on P and "$\preceq$" the following binary relation:*

$$U_1 \preceq U_2 \textit{ iff } \forall x,y \in P,\ U_1(x,y) \leq U_2(x,y).$$

Then

1. *$\langle \mathfrak{U}_P, \preceq \rangle$ is a poset;*
2. *Let $U_1, U_2 \in \mathfrak{U}_P$ be pseudo uninorms with neutral elements e_1 and e_2, respectively. If $U_1 \preceq U_2$ then $e_2 \leq e_1$;*
3. *Let $U_1, U_2 \in \mathfrak{U}_P$ be pseudo uninorms with neutral elements e_1 and e_2, respectively. If neither $e_1 \leq e_2$ nor $e_2 \leq e_1$ then neither $U_1 \preceq U_2$ nor $U_2 \preceq U_1$.*
4. *If $\langle P, \leq \rangle$ has a greater and a least element then $\langle \mathfrak{U}_P, \preceq \rangle$ also have a greater and a least element.*

Corollary 3. *Let $\langle P, \leq \rangle$ be a poset. Then,*

1. *If T is a pseudo t-norm on P then, $T(x,y) \leq x$ and $T(x,y) \leq y$.*
2. *If S is a pseudo t-conorm on P then, $x \leq S(x,y)$ and $y \leq S(x,y)$.*

It is obvious that uninorms, pseudo t-norms and pseudo t-conorms on a bounded lattice are pseudo uninorms on the same lattice. But, there exist pseudo uninorms which are neither uninorms, pseudo t-norms nor pseudo t-conorms. The following proposition provides an infinite family of such pseudo uninorms.

The following results generalize the Proposition 2.1 and 2.2 of [5].

Proposition 5. *Let $\langle P, \leq \rangle$ be a join-semilattice with top element and $T : P \times P \to P$ be a pseudo t-norm on P. Then for any $e \in P$ and isotone bijection $\phi : P_e \to P$, the mapping $U_e : P \times P \to P$ defined by:*

$$U_e(x,y) = \begin{cases} \phi^{-1}(T(\phi(x),\phi(y))) & \text{if } x,y \in P_e \\ x \vee y & \text{if } x,y \notin P_e \\ x & \text{if } x \notin P_e \text{ and } y \in P_e \\ y & \text{if } x \in P_e \text{ and } y \notin P_e \end{cases} \tag{2}$$

is a pseudo uninorm on P with e as neutral element.

Proposition 6. *Let $\langle P, \leq \rangle$ be a meet-semilattice with bottom element and $S : P \times P \to P$ be a pseudo t-conorm on P. Then for any $e \in P$ and isotone bijection $\psi : P^e \to P$, the mapping $U^e : P \times P \to P$ defined by:*

$$U^e(x,y) = \begin{cases} \psi^{-1}(S(\psi(x),\psi(y))) & \text{if } x,y \in P^e \\ x \wedge y & \text{if } x,y \notin P^e \\ x & \text{if } x \notin P^e \text{ and } y \in P^e \\ y & \text{if } x \in P^e \text{ and } y \notin P^e \end{cases} \tag{3}$$

is a pseudo uninorm on P with e as neutral element.

Proof. Analogous to Proposition 5.

Proposition 7. *Let $\langle P, \leq \rangle$ be a poset, $e \in P$ and $U_1, U_2 \in \mathfrak{U}_P^e$. Then the mapping $U_1 \rtimes U_2 : P \times P \to P$ defined by*

$$U_1 \rtimes U_2(x,y) = \begin{cases} U_1(x,y) & \text{if } x,y \in P_e \\ U_2(x \vee e, y \vee e) & \text{if } x,y \notin P_e \\ x & \text{if } x \notin P_e \text{ and } y \in P_e \\ y & \text{if } x \in P_e \text{ and } y \notin P_e \end{cases} \tag{4}$$

is a pseudo uninorm on P with e as neutral element.

Proposition 8. *Let $\langle P, \leq \rangle$ be a poset, $e \in P$ and $U_1, U_2 \in \mathfrak{U}_P^e$. Then the mapping $U_1 \ltimes U_2 : P \times P \to P$ defined by*

$$U_1 \ltimes U_2(x,y) = \begin{cases} U_1(x \wedge e, y \wedge e) & \text{if } x,y \notin P^e \\ U_2(x,y) & \text{if } x,y \in P^e \\ x & \text{if } x \notin P^e \text{ and } y \in P^e \\ y & \text{if } x \in P^e \text{ and } y \notin P^e \end{cases} \tag{5}$$

is a pseudo uninorm on P with e as neutral element.

Proof. Analogous.

As corollary we have the following generalization of the Theorem 1 in [8] (see also Theorem 2.1 in [27]).

Proposition 9. *Let $\langle L, \leq\rangle$ be a bounded lattice and $T, S : L \times L \to L$ be a pseudo t-norm and a pseudo t-conorm on L, respectively. Then, for any $e \in L$ and isotone bijections $\phi : L_e \to L$ and $\psi : L^e \to L$, the mappings $U_1, U_2 : L \times L \to L$ defined by:*

$$U_1(x,y) = \begin{cases} \phi^{-1}(T(\phi(x), \phi(y))) & \text{if } x, y \in L_e \\ \psi^{-1}(S(\psi(x), \psi(y))) & \text{if } x, y \in L^e \\ x & \text{if } x \notin L_e \text{ and } y \in L_e \\ y & \text{if } x \in L_e \text{ and } y \notin L_e \\ (x \wedge y) \vee e & \text{otherwise} \end{cases} \tag{6}$$

$$U_2(x,y) = \begin{cases} \phi^{-1}(T(\phi(x), \phi(y))) & \text{if } x, y \in L_e \\ \psi^{-1}(S(\psi(x), \psi(y))) & \text{if } x, y \in L^e \\ x & \text{if } x \notin L^e \text{ and } y \in L^e \\ y & \text{if } x \in L^e \text{ and } y \notin L^e \\ (x \vee y) \wedge e & \text{otherwise} \end{cases} \tag{7}$$

are pseudo uninorms on L with e as neutral element.

3.1 Annihilators of pseudo uninorms

In the literature, an element a of a set A is called **an annihilator for a function** $F : A \times A \to A$, whenever "$F(a,x) = F(x,a) = a$ for each $x \in A$". For example, zero is an annihilator for the usual multiplication. It is not difficult to see that an annihilator for F is unique and also that bottom, $\perp$, and top, $\top$, elements are annihilators for pseudo t-norm and pseudo t-conorm, respectively.

From this point on, the expression $U(\perp, \top)$ will be denoted by a_U.

Theorem 1. *Let $\langle P, \leq, \perp, \top\rangle$ be a bounded poset and $e \in P$. If $U \in \mathfrak{U}_P^e$ has an annihilator, then it is a_U.*

The next proposition is a generalization of the Lemma 1 in [8].

Proposition 10. *Let $\langle P, \leq, \perp, \top\rangle$ be a bounded poset and $e \in P$. If $U \in \mathfrak{U}_P^e$, then*

1. *$U(a_U, x) \leq a_U \leq U(x, a_U)$ for all $x \in P$;*
2. *$U(a_U, x) = a_U$ for all $x \in P^e$;*
3. *$U(x, a_U) = a_U$ for all $x \in P_e$.*

Corollary 4. *Let $\langle P, \leq, \perp, \top\rangle$ be a bounded poset and $e \in P$. If $U \in \mathfrak{U}_P^e$ is commutative then U has an annihilator element.*

The next proposition is stronger, since the commutativity is relaxed whereas the existence of an annihilator is maintained.

Proposition 11. *Let $\langle P, \leq, \perp, \top\rangle$ be a bounded poset and $e \in P$. If $U \in \mathfrak{U}_P^e$ is such that $U(\perp, \top) = U(\top, \perp)$, then*

– a_U *is annihilator;*
– $a_U = \bot$ *or* $a_U = \top$ *or* a_U *incomparable with* e.

Proposition 12. *Let* $\langle P, \leq, \bot, \top\rangle$ *be a bounded poset and* $e \in P$. *If* $U \in \mathfrak{U}_P^e$ *is such that* $U(\bot, \top) = U(\top, \bot) = \bot$, *then* $\bot$ *is the annihilator of* U.

Proposition 13. *Let* $\langle P, \leq, \bot, \top\rangle$ *be a bounded poset and* $e \in P$. *If* $U \in \mathfrak{U}_P^e$ *is such that* $U(\bot, \top) = U(\top, \bot) = \top$ *then* $\top$ *is an annihilator of* U.

3.2 Idempotency

In the literature, an operation $F : A \times A \to A$ is called **idempotent** whenever for each $x \in A$, $F(x, x) = x$. In this section we will confront the notion of pseudo uninorms with such property.

Proposition 14. *Let* $\langle P, \leq\rangle$ *be a poset such that* $\langle P_e, \leq_e\rangle$ *is a meet-semilattice and* $\langle P^e, \leq^e\rangle$ *is a join-semilattice.* $U \in \mathfrak{U}_P^e$ *is idempotent iff for each* $x, y \in L$,

$$U(x,y) = \begin{cases} x \wedge y & \text{if } x, y \in P_e \\ x \vee y & \text{if } x, y \in P^e \\ U(x,y) \in [x \wedge y, x \vee y] & \text{otherwise} \end{cases}$$

Corollary 5. $\langle P, \leq\rangle$ *be a meet-semilattice with a top element, denoted by* $\top$. *Then* $U \in \mathfrak{U}_P^\top$ *is idempotent iff* $U(x, y) = x \wedge y$ *for each* $x, y \in P$.

Corollary 6. $\langle P, \leq\rangle$ *be a join-semilattice with a bottom element, denoted by* $\bot$. *Then* $U \in \mathfrak{U}_P^\bot$ *is idempotent iff* $U(x, y) = x \vee y$ *for each* $x, y \in P$.

3.3 Join morphism

In this section we show how pseudo uninorms behave with respect to distributivity over supremum or just a join morphism.

Proposition 15. *Let* $\langle P, \leq\rangle$ *be a join-semilattice. If* $U \in \mathfrak{U}_P$, *then for each* $x, y, z \in P$:

1. $U(x, y \vee z) \geq U(x, y) \vee U(x, z)$, *and*
2. $U(y \vee z, x) \geq U(y, x) \vee U(z, x)$.

Definition 4. *Let* $\langle P, \leq\rangle$ *be a join-semilattice and* $U \in \mathfrak{U}_P$ *be a pseudo uninorm.* U *is a join morphism if for each* $x, y, z \in P$,

1. $U(x, y \vee z) = U(x, y) \vee U(x, z)$, *and*
2. $U(y \vee z, x) = U(y, x) \vee U(z, x)$.

Proposition 16. *Let* $\langle P, \leq\rangle$ *be a join-semilattice and* $U \in \mathfrak{U}_P$ *be a pseudo uninorm such that:*

1. *For each* $w, x, y, z \in P$, *if* $y \leq z$ *and* $U(x, y) \leq w \leq U(x, z)$, *then there exists* $u \in P$ *such that* $U(x, u) = w$,

2. for each $w, x, y, z \in P$, if $y \leq z$ and $U(y,x) \leq w \leq U(z,x)$, then there exists $u \in P$ such that $U(u,x) = w$, and
3. for each $x, y, z \in P$, if $U(x,y) \leq U(x,z)$ or $U(y,x) \leq U(z,x)$, then $y \leq z$.

Then U is join morphism.

Proposition 17. *Let $\langle P, \leq \rangle$ be a totally ordered set. Each pseudo uninorm on P is a join morphism.*

4 Kleene algebras based on pseudo uninorms

In this section we show how Kleene algebras are built by using pseudo uninorms under some conditions. In order to achieve that we propose the notion of Kleene operator based on a pseudo uninorm:

Definition 5. *Let $\langle P, \leq \rangle$ be a join-semilattice, $e \in P$ and $U \in \mathfrak{U}_P^e$ a pseudo uninorm. A Kleene operator based on U is a function $\star : P \to P$ such that for each $x, y \in P$ satisfy:*

- ***(K1)*** $e \vee U(x, x^\star) \leq x^\star$,
- ***(K2)*** $e \vee U(x^\star, x) \leq x^\star$,
- ***(K3)*** *If* $U(x,y) \leq y$ *then* $U(x^\star, y) \leq y$, *and*
- ***(K4)*** *If* $U(y,x) \leq y$ *then* $U(y, x^\star) \leq y$.

Proposition 18. *Let $\langle P, \leq \rangle$ be a join-semilattice, $e \in P$ and $U \in \mathfrak{U}_P^e$ such that either U is a join morphism or $e \in \Delta_P$. If $U(x,x) \leq x$ for each $x \in P^e$ then the operator $x^\star = x \vee e$ is a Kleene operator for U.*

Proof. Observe that $U(x,x) \leq x$ for each $x \in P_e$. So, the condition "$U(x,x) \leq x$ for each $x \in P^e$" is equivalent to "$U(x,x) \leq x \vee e$ for each $x \in P$". Let $x \in P$, then

- **(K1)** Since $x \leq x^\star$ and $x^\star \in P^e$ then $e \vee U(x, x^\star) \leq e \vee U(x^\star, x^\star) \leq e \vee x^\star = x^\star$.
- **(K2)** Analogous to **(K1)**.
- **(K3)** If U is a join morphism and $U(x,y) \leq y$ then $U(x^\star, y) = U(x \vee e, y) = U(x,y) \vee U(e,y) = U(x,y) \vee y = y$. On the other hand, if $e \in \Delta_P$ and $U(x,y) \leq y$ then when $x \leq e$ we have that $U(x^\star, y) = U(x \vee e, y) = U(e,y) = y$ and when $e \leq x$ we have that $U(x^\star, y) = U(x \vee e, y) = U(x,y) \leq y$. Therefore, in both cases, the operator $\star$ satisfy **(K3)**.
- **(K4)** Analogous to **(K3)**.

Theorem 2. *Let $\langle P, \leq \rangle$ be a join-semilattice, $e \in P$ and $U \in \mathfrak{U}_P^e$ such that either U is a join morphism or $e \in \Delta_P$. Then the operator $x^\star = x \vee e$ is a Kleene operator for U iff for each $x, y \in P^e$, $U(x,y) = x \vee y$*

Theorem 3. *Let $\langle P, \leq, \bot, \top \rangle$ be a bounded join-semilattice, $e \in \Delta_P$ and $U \in \mathfrak{U}_P^e$ such that $U(\bot, \top) = U(\top, \bot) = \bot$ and $U(x,y) = x \vee y$ for each $x, y \in P^e$. Then $\langle P, \vee, U, \star, \bot, e \rangle$ where $x^\star = x \vee e$, is a Kleene algebra.*

Theorem 4. *Let $\langle P, \leq, \bot, \top\rangle$ be a bounded join-semilattice, $e \in P$, $U \in \mathfrak{U}_P^e$ be a join morphism such that $U(\bot, \top) = U(\top, \bot) = \bot$ and $U(x,x) \leq x$ for each $x \in P^e$. Then $\langle P, \vee, U, \star, \bot, e\rangle$ where $x^\star = x \vee e$, is a Kleene algebra.*

Theorem 5. *Let $\langle K, +, U, \star, 0, e\rangle$ be a Kleene algebra. Then*

1. *$U \in \mathfrak{U}_P^e$;*
2. *$e \leq x^\star$, for each $x \in K$;*
3. *$x^\star \geq e + x$, for each $x \in K$;*
4. *$\star$ is a closure operator on $\langle K, \leq\rangle$.*

4.1 Automorphisms on $[0,1]$ acting on Kleene algebras

In fuzzy logic, a typical way of generating newer fuzzy connectives (t-norms, t-conorms and implications) from a fuzzy connective of the same type is obtained via automorphisms on the real unit interval $[0,1]$, which are defined as bijective functions on $[0,1]$ preserving natural ordering. Formally, a function $\phi : [0,1] \to [0,1]$ is an **automorphism** on $[0,1]$ if it is bijective and isotone, i.e. $x \leq y \ \Rightarrow\ \phi(x) \leq \phi(y)$ [1, 13]. In [2] is considered the equivalent definition where automorphisms are continuous and strictly isotone function satisfying the boundary conditions $\phi(0) = 0$ and $\phi(1) = 1$.

Is clear that this notion can be generalize for arbitraries posets.

Definition 6. *A function $\phi : P \to P$ is an automorphism on a poset $\langle P, \leq\rangle$ if it is bijective and for each $x, y \in P$ we have that*

$$\phi(x) \leq \phi(y) \text{ if and only if } x \leq y. \tag{8}$$

We will denote the set of all automorphism on $\langle P, \leq\rangle$ by $Aut\langle P, \leq\rangle$.

Remark 5. Let $\phi, \psi \in Aut\langle P, \leq\rangle$.

1. The inverse of an automorphism is also an automorphism. In fact, the inverse of bijection also is a bijection and $x \leq y$ iff $\phi(\phi^{-1}(x)) \leq \phi(\phi^{-1}(y))$ iff $\phi^{-1}(x) \leq \phi^{-1}(y)$.
2. The composition of two automorphism is also an automorphism. In fact, the composition of bijective functions also is bijective and $x \leq y$ iff $\psi(x) \leq \psi(y)$ iff $\phi \circ \psi(x) \leq \phi \circ \psi(y)$.
3. The identity function Id_P on P is an automorphism. In addition, $\phi \circ Id_P = \phi = Id_P \circ \phi$.

Therefore, $\langle Aut\langle P, \leq\rangle, \circ\rangle$ is a group.

Let ϕ be an automorphism on P and $f : P^n \to P$. In algebra has been extensively study the actions of groups in order to interpret the elements of the group as "acting" on some space, but preserving the structure of that space [10, 20]. Here we study the action of the group $\langle Aut\langle P, \leq\rangle, \circ\rangle$ on pseudo uninorms, Kleene operators and Kleene algebras. In general, the **action of ϕ on a function** $f : P^n \to P$, denoted by f^ϕ, is defined as follows

$$f^\phi(x_1, \ldots, x_n) = \phi^{-1}(f(\phi(x_1), \ldots, \phi(x_n))). \tag{9}$$

In particular, the action of automorphism preserve the usual fuzzy connectives [1, 2, 19] and also pseudo uninorms on $[0, 1]$ [5, Theorem 3.1]. Here we generalize this last result by consider pseudo uninorms on an arbitrary poset $\langle P, \leq\rangle$.

Proposition 19. *Let U be a pseudo uninorm and ϕ be an automorphism on a poset $\langle P, \leq\rangle$. Then U^ϕ is also a pseudo uninorm. In addition,*

1. *if $\langle P, \leq\rangle$ is bounded (with $\bot$ and $\top$ as least and great elements) then $\phi(\bot) = \bot$ and $\phi(\top) = \top$.*
2. *if $\langle P, \leq\rangle$ is a join (meet) semilattice then $\phi(x \vee y) = \phi(x) \vee \phi(y)$ ($\phi(x \wedge y) = \phi(x) \wedge \phi(y)$), i.e. ϕ is a join (meet) morphism.*

Proposition 20. *Let $\langle P, \leq\rangle$ be a join-semilattice, $e \in P$, $U \in \mathfrak{U}^e_P$ and $\phi \in Aut\langle P, \leq\rangle$. If $\star : P \to P$ is a Kleene operator for U then $\circledast : P \to P$, defined by $x^\circledast = \phi^{-1}(\phi(x)^\star)$ is a Kleene operator based on U^ϕ.*

Proposition 21. *Let $\langle K, +, \cdot, \star, 0, 1\rangle$ be a Kleene algebras and $\phi \in Aut\langle K, \leq\rangle$ where $\leq$ is the partial order defined in Equation 1. Then $\langle K, +^\phi, \cdot^\phi, \circledast, 0, 1\rangle$ also is Kleene algebra. In addition, for each $x, y \in K$ we have that $\phi(x+y) = \phi(x)+\phi(y)$, $\phi(x \cdot y) = \phi(x) \cdot \phi(y)$, $\phi(0) = 0$ and $\phi(1) = 1$.*

5 Final remarks

In this paper we have shown the relation between the notions of Kleene algebras and pseudo uninorms. We have shown that every Kleene algebra induces a pseudo uninorm and that *some* pseudo uninorms induce Kleene algebras. This connection enables both: (1) another viewpoint on the theory of Kleene algebras and (2) indicates a way to build Kleene algebras in the fuzzy setting — since we provide the requirements to build Kleene algebras from pseudo uninorms.

References

1. B. Bedregal and I. Mezzomo. Ordinal sums and multiplicative generators of the De Morgan triples. *Journal of Intelligent & Fuzzy Systems*, v. 34(4), p. 2159–2170, 2018.
2. H. Bustince, P. Burillo and F. Soria. Automorphisms, negations and implication operators. *Fuzzy Sets and Systems*, v. 134, p. 209–229, 2003.
3. C. Calk. Coherent Confluence in Modal n-Kleene Algebras. In Proc. of 9th International Workshop on Confluence 30th June 2020, Paris, France.
4. J. Conway. Regular algebra and finite machines. London, *Chapman and Hall*, 1971.
5. I.A. Da Silva, B. Bedregal, C.G. Da Costa, E.S. Palmeira and M.P. Da Rocha. Pseudo uninorms and Atanassov's intuitionistic pseudo uninorms. *Journal of Intelligent & Fuzzy Systems*, v. 29, p. 267–281, 2015.

6. J. Dombi. Basic concepts for a theory of evaluation: the aggregative operator. *Europ. J. Oper. Research*, v. 10, p. 282–293, 1982.
7. P. Flondor, G. Georgescu and A. Iorgulescu. Pseudo-t-norms and pseudo-BL algebras. *Soft Computing*, v. 5, n. 5, p. 355–371, 2001.
8. J.C. Fodor, R.R. Yager and A. Rynaloz. Structure of Uninorms. *Int. J. of Uncertainty, Fuzziness and Knowledge-Based Systems*, v. 5, n. 4, p. 411–427, 1997.
9. S. Gottwald. *Fuzzy sets and fuzzy logic*. Friedrich Vieweg & Sohn Verlag, Wiesbaden, Germany, 1993.
10. T.W. Hungerford. *Algebra*. Graduate Texts in Mathematics, Springer-Verlag, Heildelberg, 2000.
11. S. Jenei. A survey on left-continuous t-norms and pseudo t-norms. In: *Logical, Algebraic, Analytic, and Probabilistic Aspects of Triangular Norms*. E.P. Klement and R. Mesiar (Eds.). Amsterdam: Elsevier, 2005.
12. M. Kalina. Uninorms and nullnorms and their idempotent versions on bounded posets. In: Torra V., Narukawa Y., Pasi G., Viviani M. (eds) Modeling Decisions for Artificial Intelligence. MDAI 2019. Lecture Notes in Computer Science, vol 11676. Springer, Cham. https://doi.org/10.1007/978-3-030-26773-5_12
13. E. Klement and M. Navara. A survey on different triangular norm-based fuzzy logics. *Fuzzy Sets and Systems*, v. 101, p. 241–251, 1999.
14. D. Kozen. On Kleene algebras and closed semirings. In Proc. 15th Symp., MFCS'90, Banská Bystrica/Czech. 1990 (PDF). *Lecture Notes Computer Science* v. 452, Springer-Verlag, p. 26–47, 1990.
15. H.W. Liu. Two classes of pseudo-triangular norms and fuzzy implications. *Computer and mathematics with Applications*, vol. 61, p. 783–789, 2011.
16. M. Luo and N. Yao. Some extensions of the logic psUL. H. Deng et al. (Eds.): AICI 2011, Part I, LNAI 7002, p. 609–617, 2011.
17. M. Mas, M. Monserrat and J. Torrens. On Left and Right Uninorms. *Int. J. of Uncertainty, Fuzziness and Knowledge-Based Systems* vol. 9, n. 4, p. 491–508, 2001.
18. K. Menger. Statical metrics. *Proc. of the National Academy of Sciences of the United States of America*, v. 28, p. 535–537, 1942.
19. R.H.S. Reiser, B. Bedregal and M. Baczyński. Aggregating fuzzy implications. *Information Sciences*, v. 253, p.126–146, 2013.
20. J. Rotman. *An Introduction to the Theory of Groups*. Graduate Texts in Mathematics 148. Springer-Verlag, Heildelberg, 4th edition, 1995.
21. W. Sander. Associative aggregation operators. In: T. Calvo, G. Mayor and R. Mesiar (Eds.), *Aggregation Operators: New trends and applications*, Physica–Verlag, Heidelberg, 2002.
22. R. Santiago and B. Bedregal and A. Madeira and M. Martins. On interval dynamic logic: Introducing quasi-action lattices. *Science of Computer Programming*, v. 175, p.1-16, 2019.
23. B. Schweizer and A. Sklar. Espaces Metriques aléatoires. *C.R. Acad. Sci. Paris*, v. 247, p. 2092–2094, 1958.
24. B. Schweizer and A. Sklar. Associative functions and statistical triangle inequalities. *Publ. Math. Debrecen*, v. 8, p. 169–186, 1963.
25. Y. Su and Z.D. Wang. pseudo uninorms and coimplications on a complete lattice. *Fuzzy Sets and Systems*, v. 224, p. 53–62, 2013.
26. S.M. Wang. Logics for residuated pseudo uninorms and their residua. *Fuzzy Sets and Systems*, v. 218, p. 24–31, 2013.
27. Z.D. Wang and J.X. Fang. Residual operations of left and right uninorms on a complete lattice. *Fuzzy Sets and Systems*, v. 160 (1), p. 22–31, 2009.

28. Z.D. Wang, and J.X. Fang. Residual coimplicators of left and right uninorms on a complete lattice. *Fuzzy Sets and Systems*, v. 160 (14), p. 2086–2096, 2009.
29. Z.D. Wang and Y.D. Yu. Pseudo-t-norms and implication operators on a complete Brouwerian lattice. *Fuzzy Sets and Systems*, v. 132, p. 113–124, 2002.
30. R.R. Yager and A. Rybalov. Uninorm aggregation operators. *Fuzzy Sets and Systems*, v. 80, p. 111–120, 1996.

Appendix with proofs

Proof of Proposition 4

Proof. 1. Straightforward.
2. $e_2 = U_1(e_2, e_1) \le U_2(e_2, e_1) = e_1$.
3. If $U_1 \le U_2$ then by previous item $e_2 \le e_1$. Analogously, if $U_2 \le U_1$ then by previous item $e_1 \le e_2$. Therefore, if e_1 and e_2 are not comparable, then also are not comparable U_1 with U_2.
4. Let $\perp$ and $\top$ be the least and the greater element of P, respectively. Then, let

$$U_\top(x,y) = \begin{cases} x & \text{if } y = \perp \\ y & \text{if } x = \perp \\ \top & \text{otherwise} \end{cases} \qquad U_\perp(x,y) = \begin{cases} x & \text{if } y = \top \\ y & \text{if } x = \top \\ \perp & \text{otherwise} \end{cases}$$

It is easy to prove that $U_\perp, U_\top \in \mathfrak{U}_P$. Let $U \in \mathfrak{U}_P$ with $e \in P$ as neutral element and $x, y \in P$. Then if $x = \top$ then $U_\perp(\top, y) = y = U(e, y) \le U(\top, y)$. Analogously, for the case $y = \top$. Now, if $x \ne \top$ and $y \ne \top$ then $U_\perp(x, y) = \perp \le U(x, y)$. So, $U_\perp \le U$. Analogously, it is proven that $U \le U_\top$.

Proof of Proposition 5

Proof. Let $x, y, z \in P$. If $x, y, z \in P_e$, then

$$\begin{aligned} U_e(x, U_e(y,z)) &= \phi^{-1}(T(\phi(x), T(\phi(y), \phi(z)))) \\ &= \phi^{-1}(T(T(\phi(x), \phi(y)), \phi(z))) \\ &= U_e(U_e(x,y), z). \end{aligned}$$

In any other case, $U_e(x, U_e(y,z)) = \vee\{x,y,z\} \cap \overline{P_e} = U_e(U_e(x,y),z)$. Therefore, U_e is associative.

Let $x \in P$. If $x \in P_e$, then $U_e(x,e) = \phi^{-1}(T(\phi(x),\phi(e))) = x = \phi^{-1}(T(\phi(e),\phi(x))) = U_e(e,x)$. If $x \notin P_e$ then $U_e(x,e) = x = U_e(e,x)$. Therefore e is a neutral element of U_e.

Let $x, y, z \in P$ such that $y \le z$. If $x \notin P_e$ we have the following cases:

1. $y \notin P_e$: then $z \notin P_e$ and therefore, $U_e(x,y) = x \vee y \le x \vee z = U_e(x,z)$.
2. $z \in P_e$: then $y \in P_e$ and therefore, $U_e(x,y) = x = U_e(x,z)$.
3. $y \in P_e$ and $z \notin P_e$ and therefore, $U_e(x,y) = x \le x \vee z = U_e(x,z)$.

If $x \in P_e$ the we have three cases:

1. $y \notin P_e$: then $z \notin P_e$ and therefore,
   $U_e(x,y) = y \le z = U_e(x,z)$.
2. $z \in P_e$: then $y \in P_e$ and therefore,
   $U_e(x,y) = \phi^{-1}(T(\phi(x),\phi(y))) \le \phi^{-1}(T(\phi(x),\phi(z))) = U_e(x,z)$.
3. $y \in P_e$ and $z \notin P_e$ and therefore,
   $U_e(x,y) = \phi^{-1}(T(\phi(x),\phi(y))) \le y \le z = U_e(x,z)$.

Therefore, U_e is isotone in the second component. The prove that is isotone in the first component is analogous.

Proof of Proposition 7

Proof. Let $x, y, z \in P$ such that $y \leq z$. If $x \notin P_e$ then:

- Case $z \in P_e$: then $y \in P_e$ and therefore $U_1 \rtimes U_2(x, y) = x = U_1 \rtimes U_2(x, z)$.
- Case $y, z \notin P_e$: then $U_1 \rtimes U_2(x, y) = U_2(x \vee e, y \vee e) \leq U_2(x \vee e, z \vee e) = U_1 \rtimes U_2(x, z)$.
- Case $y \in P_e$ and $z \notin P_e$: then, by Remark 2, $U_1 \rtimes U_2(x, y) = x \leq x \vee e \leq U_2(x \vee e, z \vee e) = U_1 \rtimes U_2(x, z)$.

If $x \in P_e$ then:

- Case $y, z \in P_e$: Then we have that $U_1 \rtimes U_2(x, y) = U_1(x, y) \leq U_1(x, z) = U_1 \rtimes U_2(x, z)$.
- Case $y \in P_e$ and $z \notin P_e$: Then, by Remark 2, we have that $U_1 \rtimes U_2(x, y) = U_1(x, y) \leq y \leq z = U_1 \rtimes U_2(x, z)$.
- Case $y \notin P_e$: then $z \notin P_e$ and therefore $U_1 \rtimes U_2(x, y) = y \leq z = U_1 \rtimes U_2(x, z)$.

Therefore, $U_1 \rtimes U_2$ is isotone in the second component. The prove that is isotone in the first component is analogous.

Let $x, y, z \in P$. Case $x, y, z \in P_e$ or $x, y, z \notin P_e$ we have that by associativity of U_1 and U_2, $U_1 \rtimes U_2(x, U_1 \rtimes U_2(y, z)) = U_1 \rtimes U_2(U_1 \rtimes U_2(x, y), z)$. The other cases:

1. Case $x, y \in P_e$ and $z \notin P_e$: $U_1 \rtimes U_2(x, U_1 \rtimes U_2(y, z)) = z = U_1 \rtimes U_2(U_1 \rtimes U_2(x, y), z)$.
2. Case $x \in P_e$ and $y, z \notin P_e$: $U_1 \rtimes U_2(x, U_1 \rtimes U_2(y, z)) = U_2(y \vee e, z \vee e) = U_1 \rtimes U_2(U_1 \rtimes U_2(x, y), z)$.
3. Case $x, y \notin P_e$ and $z \in P_e$: $U_1 \rtimes U_2(x, U_1 \rtimes U_2(y, z)) = U_2(x \vee e, y \vee e) = U_1 \rtimes U_2(U_1 \rtimes U_2(x, y), z)$.
4. Case $x \notin P_e$ and $y, z \in P_e$: $U_1 \rtimes U_2(x, U_1 \rtimes U_2(y, z)) = x = U_1 \rtimes U_2(U_1 \rtimes U_2(x, y), z)$.

Therefore, $U_1 \rtimes U_2$ is associative, and since e is clearly a neutral element then $U_1 \rtimes U_2$ is a pseudo uninorm.

Proof of Proposition 9

Proof. By Propositions 5 and 6, we have that U_e and U^e are pseudo uninorm with e as neutral element. So, by Proposition 7, $U_e \rtimes U^e$ also is pseudo uninorm with e as neutral element. Therefore, since

$$
\begin{aligned}
U_e \rtimes U^e(x, y) &= \begin{cases} U_e(x, y) & \text{if } x, y \in L_e \\ U^e(x \vee e, y \vee e) & \text{if } x, y \notin L_e \\ x & \text{if } x \notin L_e \text{ and } y \in L_e \\ y & \text{if } x \in L_e \text{ and } y \notin L_e \end{cases} \\
&= \begin{cases} \phi^{-1}(T(\phi(x), \phi(y))) & \text{if } x, y \in L_e \\ \psi^{-1}(S(\psi(x), \psi(y))) & \text{if } x, y \in L^e \\ x & \text{if } x \notin L_e \text{ and } y \in L_e \\ y & \text{if } x \in L_e \text{ and } y \notin L_e \\ (x \wedge y) \vee e & \text{otherwise} \end{cases} \\
&= U_1(x, y)
\end{aligned}
$$

then U_1 is a pseudo uninorm with e as neutral element.
Analogouysly is possible to prove that $U_2 = U_e \ltimes U^e$ and therefore, by Proposition 8, pseudo uninorm with e as neutral element.U_2 also is a pseudo uninorm with e as neutral element.

Proof of Proposition 10

Proof. Let $x \in P$, then $U(a_U, x) \leq U(a_U, \top) = U(U(\bot, \top), \top) = U(\bot, U(\top, \top)) = U(\bot, \top) = a_U$ and $U(x, a_U) \geq U(\bot, U(\bot, \top)) = U(U(\bot, \bot), \top) = U(\bot, \top) = a_U$. Therefore, $U(a_U, x) \leq a_U \leq U(x, a_U)$ for all $x \in P$.

If $x \geq e$ then $U(\top, x) \geq U(\top, e) = \top$ and so $U(\top, x) = \top$. Therefore, $U(a_U, x) = U(U(\bot, \top), x) = U(\bot, U(\top, x)) = U(\bot, \top) = a_U$ and $U(x, a_U) = U(x, U(\bot, \top)) = U(U(x, \bot), \top) \geq U(\bot, \top) = a_U$.

If $x \leq e$ then $U(x, \bot) \leq U(e, \bot) = \bot$ and so $U(x, \bot) = \bot$. Therefore, $U(x, a_U) = U(x, U(\bot, \top)) = U(U(x, \bot), \top) = U(\bot, \top) = a_U$ and $U(a_U, x) = U(U(\bot, \top), x) = U(\bot, U(\top, x)) \leq U(\bot, \top) = a_U$. Hence, $U(x, a_U) = a_U \geq U(a_U, x)$.

Proof of Proposition 11

Proof. Let $x \in P$. Then, $U(a_U, x) \geq U(a_U, \bot) = U(U(\bot, \top), \bot) = U(U(\top, \bot), \bot) = U(\top, U(\bot, \bot)) = U(\top, \bot)$ and $U(x, a_U) \leq U(\top, U(\bot, \top)) = U(\top, U(\top, \bot)) = U(U(\top, \top), \bot) = U(\top, \bot)$. Therefore, by Proposition 10, $U(\top, \bot) \leq U(a_U, x) \leq a_U \leq U(x, a_U) \leq U(\top, \bot) = U(\top, \bot)$ and, consequently, $U(a_U, x) = a_U = U(x, a_U)$. Hence, a_U is an annihilator of U.

If $a_U \leq e$ then $a_U = U(\bot, a_U) \leq U(\bot, e) = \bot$ and so $a_U = \bot$. If $a_U \geq e$ then $a_U = U(\top, a_U) \geq U(\top, e) = \top$ and so $a_U = \top$. Therefore, $a_U = \bot$ or $a_U = \top$ or a_U incomparable with e.

Proof of Proposition 12

Proof. Let $x \in P$. Since U is isotone then $U(\bot, x) \leq U(\bot, \top) = \bot$ and $U(x, \bot) \leq U(\top, \bot) = \bot$. Therefore, $U(x, \bot) = U(\bot, x) = \bot$ for each $x \in P$.

Proof of Proposition 13

Proof. Let $x \in P$. Since U is isotone then $U(x, \top) \geq U(\bot, \top) = \top$ and $U(\top, x) \geq U(\top, \bot) = \top$. Therefore, $U(x, \top) = U(\top, x) = \top$ for each $x \in P$.

Proof of Proposition 14

Proof. ($\Rightarrow$) If $x, y \in P_e$ then, by one hand, $U(x, y) \leq U(x, e) = x$ and $U(x, y) \leq U(e, y) = y$ and therefore, $U(x, y) \leq x \wedge y$. On the other hand, $x \wedge y = U(x \wedge y, x \wedge y) \leq U(x, y)$. Therefore, $U(x, y) = x \wedge y$.

If $x, y \in P^e$ then, by one hand, $U(x, y) \geq x \vee y$ and by the other hand, $x \vee y = U(x \vee y, x \vee y) \geq U(x, y)$. Therefore, $U(x, y) = x \vee y$.

In other case:

– If x and y are comparable, then by a symmetric argument it is sufficient to consider the case $x \in P_e$ and $y \in P^e$, and therefore $x \leq y$. Thereby, $x = U(x,x) \leq U(x,y)$ and $U(x,y) = U(U(x,x),y) = U(x,U(x,y)) \leq U(x,e) = x$, i.e. $U(x,y) = x \wedge y$.
– If x and y are not comparable, then $x \in P_e$ and $y \notin P_e \cup P^e$, or, $x \in P^e$ and $y \notin P_e \cup P^e$. In the first case, $U(x,y) \leq U(e,y) = y \leq x \vee y$ and $U(x,y) \geq U(x, y \wedge e) = x \wedge y \wedge e = x \wedge y$. Analogously, in the second case, $U(x,y) \geq U(e,y) = y \geq x \wedge y$ and $U(x,y) \leq U(x, y \vee e) = x \vee y \vee e = x \vee y$. So, in both cases, $U(x,y) \in [x \wedge y, x \vee y]$.

($\Leftarrow$) Straightforward.

Proof of Proposition 15

Proof. For each $x,y,z \in P$ we have that once $U(x,y) \leq U(x, y \vee z)$ and $U(x,z) \leq U(x, y \vee z)$ then $U(x,y) \vee U(x,z) \leq U(x, y \vee z)$. The prove that $U(y \vee z, x) \geq U(y,x) \vee U(z,x)$ is analogous.

Proof of Proposition16

Proof. By Proposition 15,

$$U(x,y) \vee U(x,z) \leq U(x, y \vee z) \tag{10}$$

Since, $U(x,y) \leq U(x,y) \vee U(x,z)$ and $U(x,z) \leq U(x,y) \vee U(x,z)$ then by property 1., there exist $u \in P$ such that $U(x,u) = U(x,y) \vee U(x,z)$ and therefore, by Eq. (10), $U(x,u) \leq U(x, y \vee z)$. So, by property 3., $u \leq y \vee z$. Thus, because $u \geq y$ and $u \geq z$, we have that $u = y \vee z$ and consequently $U(x,u) = U(x, y \vee z)$. Hence, $U(x,y) \vee U(x,z) = U(x,u) = U(x, y \vee z)$.

The prove that $U(y \vee z, x) = U(y,x) \vee U(z,x)$ is analogous.

Proof of Proposition 17

Proof. Let $x,y,z \in P$. Since, P is totally ordered, by a symmetric argument, it is sufficient just consider that $y \leq z$. So, $U(x,y) \leq U(x,z)$ and $U(y,x) \leq U(z,x)$. Therefore, $U(x,y) \vee U(x,z) = U(x,z) = U(x, y \vee z)$ and $U(y,x) \vee U(z,x) = U(z,x) = U(y \vee z, x)$.

Proof of Theorem 3

Proof. The axioms **(KA1)** to **(KA4)** follows from definition of join-semilattice and least element, the axioms **(KA5)** and **(KA6)** from definition of pseudo uninorm, the axiom **(KA9)** from Proposition 12, and the axioms **(KA10)** to **(KA13)** from Proposition 18. Let $x,y,z \in P$. Then, since $e \in \Delta_P$, we have the following cases:

1. Case $x,y,z \in P^e$ then, from Theorem 2, we have that $U(x, y \vee z) = x \vee (y \vee z) = (x \vee y) \vee (x \vee z) = U(x,y) \vee U(x,z)$.

2. Case $y \in P_e$ and $z \in P^e$ then $y \leq z$ and therefore $U(x, y \vee z) = U(x, z) = U(x, y) \vee U(x, z)$.
3. Case $y \in P^e$ and $z \in P_e$ then $z \leq y$ and therefore $U(x, y \vee z) = U(x, y) = U(x, y) \vee U(x, z)$.
4. Case $y, z \in P_e$ then $U(x, y \vee z) \leq U(x, y \vee e) = U(x, y)$ and $U(x, y \vee z) \leq U(x, z \vee e) = U(x, z)$ and therefore, $U(x, y \vee z) \leq U(x, y) \vee U(x, z)$. So, because, trivially $U(x, y) \vee U(x, z) \leq U(x, y \vee z)$, then $U(x, y) \vee U(x, z) = U(x, y \vee z)$.

Therefore, the axiom **(KA7)** is satisfied for each $x, y, z \in P$. The axiom **(KA8)** can be proved in analogous way.

Proof of Theorem 4

Proof. The axioms **(KA1)** to **(KA4)** follows from definition of join-semilattice and least element, the axioms **(KA5)** and **(KA6)** from definition of pseudo uninorm,the axioms **(KA7)** and **(KA8)** because U is a join morphism, the axiom **(KA9)** from Proposition 12, and the axioms **(KA10)** to **(KA13)** from Proposition 18.

Proof of Theorem 5

Proof. 1. By the axioms **(KA5)** and **(KA6)**, U is associative and e is a neutral element. Suppose that $y \leq z$ then $U(x, z) = U(x, y + z) = U(x, y) + U(x, z)$ and therefore $U(x, y) \leq U(x, z)$. Analogously is proved that if $x \leq y$, then $U(x, z) \leq U(y, z)$.
2. Let $x \in K$. Then by **(KO1)** and **(KO2)**, once $0 \leq x$ we have that $e \leq x^\star$.
3. If $x \nleq e$ then by **(KO3)**, $U \in \mathfrak{U}_P^e$ and by previous item, $x^\star = e + U(x, x^\star) \geq e + U(x, e) = e + x$.
   Now, if $x \leq e$ then $U(x, e) \leq e$. So, by **(KA12)**, we have that $x^\star = U(x^\star, e) \leq e$. But, once by previous item $e \leq x^\star$, then $x^\star = e$. So, if $x \leq e$ then $x^\star = e = e + x$.
4. **(C1)** follows from **(KO1)**, **(C3)** follows from **(KO4)** and **(C2)** follows from previous item. In fact, $x \leq x + e \leq x^\star$.

Proof of Proposition 19

Proof. Associativity: Let $x, y, z \in P$. Then, by equations (9) and (8), and the associativity of U,

$$\begin{aligned}
&U^\phi(U^\phi(x, y), z)\\
&= \phi^{-1}(U(\phi(\phi^{-1}(U(\phi(x), \phi(y)))), \phi(z)))\\
&= \phi^{-1}(U(U(\phi(x), \phi(y)), \phi(z)))\\
&= \phi^{-1}(U(\phi(x), U(\phi(y)), \phi(z)))\\
&= \phi^{-1}(U(\phi(x), \phi(\phi^{-1}(U(\phi(y)), \phi(z)))))\\
&= U^\phi(x, U^\phi(y, z))
\end{aligned}$$

Isotonicity: Let $x_1, x_2, y_1, y_2 \in P$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$. Then by equation (9), Equation (8) and isotonicity of U,

$$\begin{aligned}
U^\phi(x_1, y_1) &= \phi^{-1}(U(\phi(x_1), \phi(y_1)))\\
&\leq \phi^{-1}(U(\phi(x_2), \phi(y_2)))\\
&= U^\phi(x_2, y_2)
\end{aligned}$$

Neutral element: Let $x \in P$. Then by equation (9) and the existence of neutral element for U (denoted by e), $U^{\phi}(x, \phi^{-1}(e)) = \phi^{-1}(U(\phi(x), e)) = x$. So, $\phi^{-1}(e)$ is the neutral element of U^{ϕ}.

Bound preserving: Since ϕ is bijective, there exists $y \in P$ such that $\phi(y) = \bot$. Thus, once $\bot \leq y$ then by equation (8), $\phi(\bot) \leq \bot$. Analogously, we prove that $\top \leq \phi(\top)$.

Joint (meet) morphism: Since $x \leq x \vee y$ and $y \leq x \vee y$ then $\phi(x) \leq \phi(x \vee y)$ and $\phi(y) \leq \phi(x \vee y)$. So, $\phi(x) \vee \phi(y) \leq \phi(x \vee y)$. On the other hand, since ϕ is bijective, there exists $z \in P$ such that $\phi(z) = \phi(x) \vee \phi(y)$. Therefore, $\phi(z) \geq \phi(x)$ and $\phi(z) \geq \phi(y)$. Hence, by Equation (8), $z \geq x$ and $z \geq y$, i.e. $z \geq x \vee y$, and $z \leq x \vee y$. Consequently, $\phi(x) \vee \phi(y) = \phi(x \vee y)$. The proof that ϕ is a meet morphism (when $\langle P, \leq \rangle$ is a meet-semilattice) is analagous.

Proof of Proposition 20

Proof. By Proposition 19, $U^{\phi} \in \mathfrak{U}_P^{\phi^{-1}(e)}$.

(K1) Since, $\star$ is Klene opertaor for U and φ^{-1} is an automorphism and therefore is a join morphism and isotone, $\phi^{-1}(e) \vee U^{\phi}(x, x^{\circledast}) = \phi^{-1}(e) \vee \phi^{-1}(U(\phi(x), \phi(x)^{\star})) =$
$\phi^{-1}(e \vee U(\phi(x), \phi(x)^{\star})) \leq \phi^{-1}(\phi(x)^{\star}) = x^{\circledast}$.

(K2) Analogous to **(K1)**.

(K3) If $U^{\phi}(x, y) \leq y$ then $U(\phi(x), \phi(y)) \leq \phi(y)$. So, because $\star$ is a Kleene operator based on U, $U(\phi(x)^{\star}, \phi(y)) \leq \phi(y)$. Therefore, because $\phi^{-1} \in Aut\langle P, \leq \rangle$, we have that $U^{\phi}(x^{\circledast}, y) \leq y$.

(K4) Analogous to **(K3)**.

Proof of Proposition 21

Proof. **(KA1)** $a +^{\phi} (b +^{\phi} c) = \phi^{-1}(\phi(a) + (\phi(b) + \phi(c))) = \phi^{-1}((\phi(a) + \phi(b)) + \phi(c)) = (a +^{\phi} b) +^{\phi} c$.

(KA2) $a +^{\phi} b = \phi^{-1}(\phi(a) + \phi(b)) = \phi^{-1}(\phi(b) + \phi(a)) = b +^{\phi} a$.

(KA3) $a +^{\phi} a = \phi^{-1}(\phi(a) + \phi(a)) = \phi^{-1}(\phi(a)) = a$.

(KA4) $a +^{\phi} 0 = \phi^{-1}(\phi(a) + 0) = \phi^{-1}(\phi(a)) = a$.

(KA5) Analogous to **(KA1)**.

(KA6) Analogous to **(KA4)**.

(KA7) $a \cdot^{\phi} (b +^{\phi} c) = \phi^{-1}(\phi(a) \cdot (\phi(b) + \phi(c))) = \phi^{-1}((\phi(a) \cdot \phi(b)) + (\phi(a) \cdot \phi(c))) = (a \cdot^{\phi} b) +^{\phi} (a \cdot^{\phi} c)$.

(KA8) Analogous to previous item.

(KA9) Analogous to **(KA4)**.

(KA10) $1 +^{\phi} (a \cdot^{\phi} a^{\circledast}) = \phi^{-1}(1 + (\phi(a) \cdot \phi(a)^{\star})) \leq \phi^{-1}(\phi(a)^{\star}) = a^{\circledast}$

(KA11) Analogous to previous item.

(KA12) If $a \cdot^{\phi} b \leq b$ then $\phi^{-1}(\phi(a) \cdot \phi(b)) \leq b$ and so $\phi(a) \cdot \phi(b) \leq \phi(b)$. Hence, $\phi(a)^{\star} \cdot \phi(b) \leq \phi(b)$ and therefore $a^{\circledast} \cdot^{\phi} b \leq b$.

(KA13) Analogous to previous item.

In addition, since for each $x, y \in K$, we have that $x \leq x + y$ and $y \leq x + y$ then (*) $x \vee y \leq x + y$, where $x \vee y$ is the supremum of x and y w.r.t. $\leq$. By [14] we have that if $a \leq b$ then $a + c \leq b + c$, and therefore, since $x \leq x \vee y$ then $x + y \leq (x \vee y) + y$. But, because $y \leq x \vee y$ by Equation (1) $(x \vee y) + y = x \vee y$ and therefore (**) $x + y \leq x \vee y$. Hence, from (*) and (**), $x + y = x \vee y$ for each $x, y \in K$. Analogously we can prove that $x \cdot y = x \wedge y$ for each $x, y \in K$. Consequently, because ϕ is an automorphism $\phi(0) = 0$, $\phi(1) = 1$, and it is a join and meet morphism and so $\phi(x + y) = \phi(x) + \phi(y)$ and $\phi(x \cdot y) = \phi(x) \cdot \phi(y)$.

Dynamic Epistemic Logic for Budget-Constrained Agents

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Abstract. We present a static ($\mathsf{EL_{bc}}$) and dynamic ($\mathsf{DEL_{bc}}$) epistemic logic for budget-constrained agents, in which an agent can obtain some information in exchange for budget resources. $\mathsf{EL_{bc}}$ extends a standard multi-agent epistemic logic with expressions concerning agent's budgets and formulas' costs. $\mathsf{DEL_{bc}}$ extends $\mathsf{EL_{bc}}$ with dynamic modality "$[?_i A]\varphi$" which reads as "φ holds after i's question whether a propositional formula A is true". In this paper we provide a sound and complete axiomatization for $\mathsf{EL_{bc}}$ and $\mathsf{DEL_{bc}}$ and show that both logics are decidable.

Keywords: Dynamic epistemic logic · Budget-constrained agents · Knowledge representation.

1 Introduction

Dynamic epistemic logic [6, 12, 15] is a common way of describing agents' knowledge and informational changes. But nowadays, our intuition about the nature of agents' reasoning and interaction tells us that both processes of operating with available knowledge and obtaining a new one cannot always be effortless. This natural intuition demonstrates that reasoning often becomes a resource consuming action. A lot of researchers of epistemic logic paid attention to this problem and found different approaches to formalising the idea of resource-bounded agents [10]. The wide range of existing approaches, describing non-omniscient agents, consider resources as various cognitive limits.

Non-omniscience can be described through time- and memory-constrained agents who do not necessarily know all the logical consequences of their knowledge. Some papers model such constraints through so-called inferential actions, which require agents to take explicit inference steps, spending available resources to deduce the logical consequences of their knowledge [17]. Other papers extend the idea of a bounded deliberation process with resource consuming inference actions by introducing perception [4] or rule-based models [14] and their effects on formation of agents' beliefs. The idea of resource-bounded agents, situated in agent–environment systems that takes into account agents' observations, beliefs, goals and actions, sounds promising both for philosophers and computer scientists [2]. Most contemporary papers on resource-bounded reasoning would agree that modelling of non-omniscient agents does not mean modelling of imperfect reasoners. On the contrary, a lot of papers argue that epistemic logic must formalise the idea that if the agent knows all necessary premises and either thinks hard enough [8] or has enough time [3, 1], then they will know the conclusion. Thus, the reasoning process itself can justifiably be considered as an ongoing time-consuming [13], as well as a memory-consuming [9] process. This intuition bridges the gap between reasoning and computation process and sounds fruitful for AI research. While this is a reasonable assumption which is worth studying, both time- and memory-based approaches deal with 'inner' obstacles of an agent's deliberation process. Thus, even existing papers studying resource constraints in agent-environment settings consider resources as a tool of reasoning or obtaining new information from already available agent's knowledge. At the same time, a lot of real-life scenarios demonstrate that resources can also be considered as an instrument of obtaining new, independent or already available, information from the outside. In other words, solving some tasks can require getting additional information, which is not necessarily costless. Our main goal in this paper is to consider logically omniscient reasoners who can interact with the environment (in the sense of an independent bystander) and obtain new information from this environment by spending a certain amount of resources.

A similar attempt was made by Naumov and Tao [16]. Their paper describes budget-constrained agents in epistemic settings. It catches the intuition that sometimes agents have to spend their resources to obtain the knowledge of some fact. But since their logic is static and describes resource constraints as a feature of the knowledge of the operator itself, this approach violates the Negative Introspection axiom, so it requires to be considered like a S4-like system. Nevertheless, this S4-like epistemic logic appears to be complete, with respect to S5-like structures. Our paper aims to demonstrate that reasoning about knowledge and informational change under budget constraints can be described by an S5-like system if we consider this informational change explicitly in DEL-style language.

We assume that agents can purchase information, spending some resources available to them. Intuitively, agents can ask a question" is A true?" and get a positive or negative answer. Sometimes, this question can require some resources (e.g. money). The first example that comes to mind these days obviously involves COVID-19. We can easily imagine that Agent A can be COVID-positive without knowing about it. It is also clear that Agent A can get this information by medical testing, which usually requires some amount of money, say \$20. In this situation, Agent A can buy an answer to the question 'Am I infected?' if her budget exceeds \$20. To introduce the multi-agent dimension in this example, let's assume that Agent A is a professor at some university, U. Nowadays it is common practice that professors are asked to work remotely. Imagine that our university, U can relax these restrictions and allow working on campus for those professors (agents) who can provide a negative COVID-test. It is also easy to imagine that a university can have a list of all professors who took a test (for example, this university can be in cooperation with some medical organisation). But the results of these tests are available to professors only, due to the medical privacy. Thus, if Agent A decides to take a test, she definitely obtains the result. At the same time U (1) does not know if A is infected, but it also knows that (2) 'A knows she is infected or A knows she is not infected'. But since this action requires \$20, U also knows that (3) A had at least \$20 before testing, and if U knows that A had n_1 or n_2 (where $n_1, n_2 \geq 20$), then

(4) U knows that A has $\$(n1-20)$ or $\$(n2-20)$ now. We hope that this example is clear and represents useful intuitions about resource-consuming informational updates in a multi-agent setting. Thus, we intend to model situations in which agents can spend resources in order to obtain an answer to some question. In our framework, the very fact of the question is public. i.e. every agent knows that question is asked. But the answer is private, so only one agent knows it. We also assume that resources can be understood in some abstract way, similar to the idea of utility in economics. Thus, we can consider money, effort or any other kind of agent's utility as resources in our models. We build our logic upon the standard S5 epistemic logic [12], enriched with linear inequalities described in [11] to deal with costs of the formulas and agents' budget. Then, we extend this logic with dynamic operator $[?_i A]$ combining ideas of public announcement logic [6], contingency logic with arbitrary announcement [5] and some intuitions about semi-private announcements. Section 2 of this paper deals with static epistemic logic for reasoning about costs of formulas and agent's budget. We demonstrate that this logic is sound and complete. Section 3 provides a dynamic extension of static fragment which allows us to reason about informational change for budget-constrained agents. We also state a soundness and completeness result for dynamic fragment via standard reduction argument and prove that both $\mathsf{EL_{bc}}$ and $\mathsf{DEL_{bc}}$ are decidable.

2 Epistemic Logic for Budget-Constrained Agents

Here we present the syntax and semantics of the *epistemic logic for budget-constrained agents* $\mathsf{EL_{bc}}$. In Section 3 we extend it with the dynamic operators for model updates.

2.1 Syntax

Let $\mathsf{Prop} = \{p, q, \dots\}$ be a countable set of propositional letters. Denote by $\mathcal{L}_{PL}$ the set of all propositional (non-modal) formulas defined by the following grammar (where p ranges over Prop, other connectives are defined standardly):

$$A, B \ ::= \ p \ \mid \ \neg A \ \mid \ (A \wedge B).$$

Definition 1 (The language $\mathsf{EL_{bc}}$). Let $\mathsf{Agt} = \{i, j, \dots\}$ be a finite set of agents. We fix a set of constants $\mathsf{Const} = \{c_A \mid A \in \mathcal{L}_{PL}\} \cup \{b_i \mid i \in \mathsf{Agt}\}$. It contains a constant c_A for the *cost* of each propositional formula A and a constant b_i for the *budget* of each agent i. *Formulas* of the language $\mathsf{EL_{bc}}$ are defined by the following grammar:

$$\varphi, \psi \ ::= \ p \mid (z_1 t_1 + \ldots + z_n t_n) \geq z \mid \neg\varphi \mid (\varphi \wedge \psi) \mid K_i \varphi,$$

where p ranges over Prop, $i \in \mathsf{Agt}$, $t_1, \dots, t_n \in \mathsf{Const}$ and $z_1, \dots, z_n, z \in \mathbb{Z}$.

Other Boolean connectives $\to, \vee, \leftrightarrow, \bot$ and $\top$ are defined in the standard way. The dual operator for K_i is defined as $\hat{K}_i \varphi \equiv \neg K_i \neg\varphi$. We will also use $K_i^? \varphi$ as an abbreviation for $(K_i \varphi \vee K_i \neg\varphi)$. Note that we introduce the cost c_A only for *propositional* formulas $A \in \mathcal{L}_{PL}$. The logic with costs of arbitrary epistemic formulas is left for future research. We deal with linear inequalities and use the same abbreviations as in [11]. Thus, we write $t_1 - t_2 \geq z$ for $t_1 + (-1)t_2 \geq z$, $t_1 \geq t_2$ for $t_1 - t_2 \geq 0$, $t_1 \leq z$ for

$-t_1 \geq -z$, $t_1 < z$ for $\neg(t_1 \geq z)$, and $t_1 = z$ for $(t_1 \geq z) \wedge (t_1 \leq z)$. Thus, the language $\mathsf{EL_{bc}}$ allows us to express statements such as: "$c_{p\wedge q} \geq 7$", "$b_i \geq 5$", "$2b_i = b_j$", "$K_c(b_i + b_j \geq c_{p\vee q})$" etc.

The set of *subformulas* $\mathsf{Sub}(\varphi)$ of a formula φ is defined in the standard way; note that if a constant c_A occurs in φ then we do *not* count A as a subformula of φ.

2.2 Semantics

A *model* $\mathcal{M}$ of the logic $\mathsf{EL_{bc}}$ has the components standard for the multi-modal logic $\mathbf{S5}$, namely, a non-empty set of states W, an epistemic accessibility relation $\sim_i$ for each agent $i \in \mathsf{Agt}$, and a valuation $V\colon \mathsf{Prop} \to 2^W$. Besides, a model $\mathcal{M}$ contains a function Cost that assigns to every propositional formula at each state its cost, and a function Bdg that assigns to each agent $i \in \mathsf{Agt}$ at each state $w \in W$ the available amount of resources.

Definition 2 (Kripke-style semantics).
A *model* is a tuple $\mathcal{M} = (W, (\sim_i)_{i\in\mathsf{Agt}}, \mathsf{Cost}, \mathsf{Bdg}, V)$, where

- W is a non-empty set of *states*,
- $\sim_i \subseteq (W \times W)$ is an equivalence relation for each $i \in \mathsf{Agt}$,
- $\mathsf{Cost}\colon W \times \mathcal{L}_{PL} \longrightarrow \mathbb{R}^+$ is the (non-negative) *cost* of propositional formulas,
- $\mathsf{Bdg}\colon \mathsf{Agt} \times W \longrightarrow \mathbb{R}^+$ is the (non-negative) *bugdet* of each agent at each state,
- $V\colon \mathsf{Prop} \to 2^W$ is a *valuation* of propositional variables.

Thus both the cost of a formula and the budget of an agent depend on a current state. We use $\mathsf{Bdg}_i(w)$ as an abbreviation for $\mathsf{Bdg}(i, w)$, where $i \in \mathsf{Agt}$ and $w \in W$. In order to formulate additional constraints on the function Cost, we need the following notation. Let PL be the classical propositional logic. For any propositional formulas A and B:

- A and B are called *equivalent*: $A \equiv B$ iff $\vdash_{\mathrm{PL}} A \leftrightarrow B$,
- A and B are called *similar*: $A \approx B$ iff $A \equiv B$ or $A \equiv \neg B$.

We also impose the following conditions on the function Cost:

(C1) $\mathsf{Cost}(w, \bot) = \mathsf{Cost}(w, \top) = 0$,
(C2) $A \approx B$ implies $\mathsf{Cost}(w, A) = \mathsf{Cost}(w, B)$, for all $A, B \in \mathcal{L}_{PL}$ and all $w \in W$.

Definition 3. The *truth* $\vDash$ of a formula A at a state $w \in W$ of a model $\mathcal{M}$ is defined by induction:
$\mathcal{M}, w \vDash p$ iff $w \in V(p)$,
$\mathcal{M}, w \vDash \neg\varphi$ iff $\mathcal{M}, w \nvDash \varphi$,
$\mathcal{M}, w \vDash \varphi \wedge \psi$ iff $\mathcal{M}, w \vDash \varphi$ and $\mathcal{M}, w \vDash \psi$,
$\mathcal{M}, w \vDash K_i\varphi$ iff $\forall w' \in W\colon w \sim_i w' \Rightarrow \mathcal{M}, w' \vDash \varphi$,
$\mathcal{M}, w \vDash (z_1 t_1 + \cdots + z_n t_n) \geq z$ iff $(z_1 t'_1 + \cdots + z_n t'_n) \geq z$, where for $1 \leq k \leq n$,

$$t'_k = \begin{cases} \mathsf{Cost}(w, A), & \text{for } t_k = c_A, \\ \mathsf{Bdg}_i(w), & \text{for } t_k = b_i. \end{cases}$$

We refer to the class of all models satisfying all properties mentioned above as $\mathfrak{M}$. We write $\vDash_{\mathfrak{M}} \varphi$ if the formula φ is *valid* in the class of models $\mathfrak{M}$.

2.3 Soundness and Completeness

The axiomatisation of the logic $\mathsf{EL}_{\mathsf{bc}}$ is presented in Table 1. Here, (Ineq) is the set of axioms for linear inequalities firstly described in [11] and used later for similar purposes in [17].

Table 1. Proof system for $\mathsf{EL}_{\mathsf{bc}}$

	Axioms:
(Taut)	All instances of propositional tautologies
(Ineq)	All instances of the axioms for linear inequalities
(K)	$K_i(\varphi \to \psi) \to (K_i\varphi \to K_i\psi)$
(T)	$K_i\varphi \to \varphi$
(4)	$K_i\varphi \to K_iK_i\varphi$
(5)	$\neg K_i\varphi \to K_i\neg K_i\varphi$
(Bd)	$b_i \geq 0$
($\geq_1$)	$c_A \geq 0$
($\geq_2$)	$c_\top = 0$
($\geq_3$)	$c_A = c_B$ if $A \approx B$, for all formulas $A, B \in \mathcal{L}_{PL}$
	Inference rules:
(MP)	From φ and $\varphi \to \psi$, infer ψ
(Nec$_i$)	From φ infer $K_i\varphi$

Axioms (Ineq) allow us to prove all valid formulas about linear inequalities. These axioms are presented in Table 2.

Table 2. Axioms for reasoning about linear inequalities

(I1) $(a_1t_1 + \cdots + a_kt_k \geq c) \leftrightarrow (a_1t_1 + \cdots + a_kt_k + 0t_{k+1}) \geq c)$
(I2) $(a_1t_1 + \cdots + a_kt_k \geq c) \to (a_{j_1}t_{j_1} + \cdots + a_{j_k}t_{j_k} \geq c)$, where $j_1, \ldots, j_k$ is a permutation of $1, \ldots, k$
(I3) $(a_1t_1 + \cdots + a_kt_k \geq c) \wedge (a'_1t_1 + \cdots + a'_kt_k \geq c') \to$ $\to (a_1 + a'_1)t_1 + \cdots + (a_k + a'_k)t_k \geq (c + c')$
(I4) $(a_1t_1 + \cdots + a_kt_k \geq c) \leftrightarrow (da_1t_1 + \cdots + da_kt_k \geq dc)$ for $d > 0$
(I5) $(t \geq c) \vee (t \leq c)$
(I6) $(t \geq c) \to (t > d)$, where $c > d$

Theorem 1 (Soundness). $\mathsf{EL}_{\mathsf{bc}}$ *is sound w.r.t.* $\mathfrak{M}$*, i.e.,* $\vdash_{\mathsf{EL}_{\mathsf{bc}}} \varphi \;\Rightarrow\; \models_{\mathfrak{M}} \varphi$.

Proof. Straightforward.

For the completeness proof, fix an $\mathsf{EL}_{\mathsf{bc}}$-consistent formula φ. We start with the set $\Gamma = \mathsf{Sub}(\varphi)$ of all subformulas of φ. Next, let $\Gamma^+ \supseteq \Gamma$ be the smallest set of formulas such that

1. Γ^+ is closed under single negation: if $\psi \in \Gamma^+$ and ψ does not start with $\neg$, then $\neg\psi \in \Gamma^+$,
2. $(b_i \geq 0) \in \Gamma^+$, for each agent $i \in \mathsf{Agt}$ that occurs in Γ (in b_i or K_i),
3. $(c_A \geq 0) \in \Gamma^+$, for each constant c_A that occurs in Γ,
4. $(c_\top = 0) \in \Gamma^+$,
5. $c_A = c_B \in \Gamma^+$ for all constants c_A and c_B occurring in Γ such that $A \approx B$.

First, we build a finite *canonical pre-model* $\mathcal{M}^c = (W^c, (\sim_i^c)_{i \in \mathsf{Agt}}, V^c)$ by the construction similar to that used for the multi-agent logic $\mathbf{S5}$:

– W^c is the set of all maximal $\mathsf{EL}_{\mathsf{bc}}$-consistent subsets of Γ^+;
– $x \sim_i^c y$ iff, for all formulas $\psi \in \Gamma^+$, we have $K_i\psi \in x$ iff $K_i\psi \in y$;
– $w \in V^c(p)$ iff $p \in w$, for each propositional variable $p \in \Gamma$.

So far, $\mathcal{M}^c$ is a Kripke model, without the Cost^c and Bdg^c functions. Thus it remains to prove that both functions Cost^c and Bdg^c can be defined.

Since every state $w \in W^c$ is $\mathsf{EL}_{\mathsf{bc}}$-consistent, the set of all linear inequalities contained in w is satisfiable, i.e., has at least one solution. Then we can easily construct functions $\mathsf{Cost}^c(A, w)$ and $\mathsf{Bdg}_i^c(w)$ that agree with this solution: for formulas $A \in \mathcal{L}_{PL}$ such that c_A occurs in Γ^+, we put $\mathsf{Cost}^c(A, w)$ to be the real that corresponds to c_A in that solution; for other formulas $B \in \mathcal{L}_{PL}$, if $B \approx A$ for some formula A such that c_A is in Γ, then we put $\mathsf{Cost}^c(B, w) := \mathsf{Cost}^c(A, w)$. Thus we can enforce that for all $w \in W^c$ and all $A \in \mathcal{L}_{PL}$ such that c_A occurs in Γ^+ it holds that
(1) $\mathsf{Cost}^c(A, w) \geq 0$ for all formulas $A \in \mathcal{L}_{PL}$ such that c_A occurs in Γ^+, by the construction of Γ^+ and $(\geq_1)$ axiom,
(2) $\mathsf{Cost}^c(\top, w) = 0$, by the construction of Γ^+ and $(\geq_2)$ axiom,
(3) $\mathsf{Cost}^c(A, w) = \mathsf{Cost}^c(B, w)$ for all $A, B \in \mathcal{L}_{PL}$ such that $A \approx B$, by $(\geq_3)$ axiom.

Similarly, we construct Bdg^c function such that for each $w \in W^c$ and each $i \in Agt$, $\mathsf{Bdg}_i^c(w)$ agrees with existing solution of linear inequalities, contained in w. This construction is well-defined and for any $w \in W^c$ and any $i \in \mathsf{Agt}$, it holds that
(1) $\mathsf{Bdg}_i^c(w) \geq 0$ by axiom (Bd) and the construction of Γ^+,
(2) $\mathsf{Bdg}_i^c(w) \geq \mathsf{Cost}^*(A)$ iff $(b_i \geq c_A) \in w$, for all b_i, c_A in Γ.

Thus, we obtained a finite canonical model $\mathcal{M}^c = (W^c, (\sim_i^c)_{i \in \mathsf{Agt}}, \mathsf{Cost}^c, \mathsf{Bdg}^c, V^c)$. As we have already demonstrated, this model satisfies the properties (C1) and (C2). It is also clear that for all $i \in \mathsf{Agt}$, $\sim_i^c$ is an equivalence relation on W^c.

Lemma 1 (Truth Lemma). *For any $\psi \in \Gamma^+$, we have:* $\mathcal{M}^c, w \vDash \psi \iff \psi \in w$.

Proof. Induction on ψ. **Cases** for $p \in \mathsf{Prop}$ and Boolean connectives: trivial.

Case $K_i\psi$:
$\mathcal{M}^c, w \vDash K_i\psi$ iff $\forall w' : w \sim_i^* w' \Rightarrow M^c, w' \vDash \psi$ by Definition 3. $\forall w' : w \sim_i^* w' \Rightarrow M^c, w' \vDash \psi$ iff $\forall w' : w \sim_i^* w' \Rightarrow \psi \in w'$ by previous induction step. $\forall w' : w \sim_i^* w' \Rightarrow \psi \in w'$ iff $K_i\psi \in w$ by the construction of $\sim_i^*$.

Case $(z_1t_1 + \cdots + z_nt_n) \geq z$:
$\mathcal{M}^c, w \vDash (z_1t_1 + \cdots + z_nt_n) \geq z$ iff $(z_1t'_1 + \cdots + z_nt'_n) \geq z$, where $t'_1, \ldots, t'_n$ are represented by $\mathsf{Cost}^*(A)$ and $\mathsf{Bdg}^*_i(w)$ for the corresponding constants c_A and b_i that occur in $(z_1t_1 + \cdots + z_nt_n) \geq z$. By the construction of Cost^* and Bdg^*, it also holds that $(z_1t'_1 + \cdots + z_nt'_n) \geq z$ iff $(z_1t_1 + \cdots + z_nt_n) \geq z \in w$.

Theorem 2 (Completeness). $\mathsf{EL_{bc}}$ *is complete w.r.t.* $\mathfrak{M}$*, i.e.,* $\vDash_{\mathfrak{M}} \varphi$ *iff* $\vdash_{\mathsf{EL_{bc}}} \varphi$.

Proof. The right-to-left direction follows from Theorem 1. For the left-to-right direction, consider a formula φ such that $\nvdash_{\mathsf{EL_{bc}}} \varphi$. Construct a model $\mathcal{M}^c$ for $\neg\varphi$. From Lemma 1 it is clear that $\exists w \in W^*$ such that $\mathcal{M}^c, w \vDash \neg\varphi$. Then $\mathcal{M}^c, w \nvDash \varphi$. It is also clear that $\mathcal{M}^c \in \mathfrak{M}$, by the construction of $\mathcal{M}^c$, so $\nvDash_{\mathfrak{M}} \varphi$.

Here we should also mention that in $\mathsf{EL_{bc}}$ we intentionally impose as less semantic restrictions as possible to deal with the most general case. In particular, we assume that it is possible that an agent does not know her own budget. But this restriction can be imposed by adding the following axiom to $\mathsf{EL_{bc}}$:

$$(b_i = z) \to K_i(b_i = z) \tag{Kb}$$

Let $\mathfrak{M}^{\mathsf{Kb}}$ be a subclass of $\mathfrak{M}$ such that for any $w_1, w_2 \in W : w_1 \sim_i w_2 \Rightarrow Bdg_i(w_1) = Bdg_i(w_2)$. Then it is straightworfard to prove the following result.

Theorem 3 (Completeness). *The logic* $\mathsf{EL_{bc}} + \mathrm{Kb}$ *is complete with respect to* $\mathfrak{M}^{\mathsf{Kb}}$*, i.e.,* $\vDash_{\mathfrak{M}^{\mathsf{Kb}}} \varphi \quad \Leftrightarrow \quad \vdash_{\mathsf{EL_{bc}}+\mathsf{Kb}} \varphi$.

Theorem 4 (Decidability). *The satisfiability problem for* $\mathsf{EL_{bc}}$ *is decidable.*

Proof. In this proof we follow the technique similar to those from [7]. From the proof of Theorem 2 it follows that a formula φ is satisfiable iff it is satisfiable in a model $\mathcal{M} \in \mathfrak{M}$ with at most $2^{|\Gamma^+|}$ states. However, since these models include Cost and Bdg functions there are infinitely many of them. In order to restrict the set of structures to check to be finite, we will consider pseudo-models which do not have Cost and Bdg, but it is easy to check whether a corresponding functions exist. We call pseudo-models for which both Cost and Bdg exist solvable. The existence of one of such solvable pseudo-models satisfying φ will guarantee the existence of a proper model (for which Cost and Bdg are defined) that satisfies φ.

Consider a set Γ^+ defined in the proof of Theorem 2 and let a set $Sum(\varphi)$ be a set of all elements of Γ^+ of the form $\sum\limits_{k=1}^{n} z_kt_k \geq z$. For every $l \leq 2^{|\Gamma^+|}$ we consider a pseudo-model $\overline{\mathcal{M}} = (\overline{W}, \overline{\sim_i}, \overline{S}, \overline{V})$, where $\overline{W}, \overline{\sim_i}$ and $\overline{V}$ are defined in a standard way and $\overline{S}$ is defined as follows:
$\overline{S} : \overline{W} \times Sum(\varphi) \longrightarrow \{true, false\}$.

Note that there are only finitely many pseudo-models for each l. They are not models of our logic, but we can check if an element of Γ^+ holds in some states of this pseudo-model using the $\vDash'$ relation which is defined in a trivial way, except the case for $\sum\limits_{k=1}^{n} z_kt_k \geq z$:
$\overline{\mathcal{M}}, w \vDash \sum\limits_{k=1}^{n} z_kt_k \geq z$ iff $\overline{S}(w, \sum\limits_{k=1}^{n} z_kt_k \geq z) = true$.

We will consider only those pseudo-models $\overline{\mathcal{M}}$ such that $\overline{\mathcal{M}}, w \vDash' \varphi$ for some $w \in \overline{W}$. For each such $\overline{\mathcal{M}}$ we want to check whether $\overline{\mathcal{M}}$ can be extended to a structure $\mathcal{M} \in \mathfrak{M}$ of our logic. In other words, we want to check if $\overline{S}$ can be replaced by a tuple $(\mathsf{Cost}, \mathsf{Bdg})$ that agrees with $\overline{S}$ and for every $w \in W$ and every $\psi \in Sum(\varphi)$ we have $\mathcal{M}, w \vDash \psi$ iff $\overline{S}(w, \psi) = true$. It is straightforward to check that for such $\mathcal{M}$ it holds that $\mathcal{M}, w \vDash \chi$ iff $\overline{\mathcal{M}}, w \vDash' \chi$ for every $\chi \in \Gamma^+(\varphi)$. For this purpose we consider special system of linear inequalities to define $\mathsf{Cost}(A, w)$ and $\mathsf{Bdg}_i(w)$ for each $i \in Agt$ and each $w \in W$. We use the variables of the form $c_{\chi,w}$ and $b_{i,w}$ which represent the values of $\mathsf{Cost}(\chi, w)$ and $\mathsf{Bdg}_i(w)$ respectively. Now we are ready to define a system of linear inequalities:
(1) $c_{\chi,w} \geq 0$ for each $\chi \in \mathcal{L}_{PL} \cap \Gamma^+$ and $w \in W$,
(2) $b_{i,w} \geq 0$ for each $i \in Agt$ and each $w \in W$,
(3) $c_{\top,w} = 0$ for each $w \in W$,
(4) $c_{\chi,w} = c_{\chi',w}$ for each $w \in W$, where $\chi, \chi' \in \mathcal{L}_{PL} \cap \Gamma^+$ such that $\chi \approx \chi'$,
(5) $\sum\limits_{k=1}^{n} z_k t_k \geq z$, where each occurrence of c_A and b_i are replaced with $c_{A,w}$ and $b_{i,w}$ for every formula $\sum\limits_{k=1}^{n} z_k t_k \geq z$ such that $\overline{S}(w, \sum\limits_{k=1}^{n} z_k t_k \geq z) = true$,
(6) $\sum\limits_{k=1}^{n} z_k t_k < z$, where each occurrence of c_A and b_i are replaced with $c_{A,w}$ and $b_{i,w}$ for every formula $\sum\limits_{k=1}^{n} z_k t_k \geq z$ such that $\overline{S}(w, \sum\limits_{k=1}^{n} z_k t_k \geq z) = false$.

For our purposes it is sufficient to find at least one solution of such system of equations and inequalities. Note that this system is finite and the problem of solving systems of inequalities is decidable. So, given a pseudo-model we can check if this pseudo-model is solvable (by solving a corresponding system of inequalities). It is straightforward to see that if there is a solvable pseudo-model for φ, then φ is satisfiable.

The proof for other direction is trivial, since the canonical model for φ gives rise to a solvable pseudo-model with $2^{|\Gamma^+|}$ states. Then if φ is satisfiable, then there is a solvable pseudo-model for φ with $l \leq 2^{|\Gamma^+|}$ states.

We have shown that φ is satisfiable iff there is a solvable pseudo-model for φ with $l \leq 2^{|\Gamma^+|}$ states. So, we can check satisfiablity of φ examining finitely many choices of l for which there are only finitely many pseudo-models and each pseudo-model can be verified to be solvable in a finite number of steps.

3 Dynamic Epistemic Logic for Budget-Constrained Agents

The dynamic language $\mathsf{DEL_{bc}}$ extends the static language $\mathsf{EL_{bc}}$ with a dynamic operator $[?_i A]\varphi$. A formula $[?_i A]\varphi$ can be read as *"φ is true after i's question whether A is true"*.

3.1 Syntax

Definition 4. The *formulas* of $\mathsf{DEL_{bc}}$ are defined by the following grammar:

$$\varphi, \psi ::= p \mid (z_1 t_1 + \cdots + z_n t_n) \geq z) \mid \neg\varphi \mid (\varphi \wedge \psi) \mid K_i\varphi \mid [?_i A]\varphi,$$

where $p \in \mathsf{Prop}$, $A \in \mathcal{L}_{PL}$, $i \in Agt$, $t_1, \dots, t_n \in \mathsf{Const}$ and $z_1, \dots, z_n, z \in \mathbb{Z}$.

The dual operator $\langle ?_i A \rangle \varphi$ can be defined in a standard way: $\langle ?_i A \rangle \varphi \equiv \neg [?_i A] \neg \varphi$.

3.2 Semantics

The main features of the operator $[?_i A]\varphi$ are: (1) every agent knows that the question was asked, i.e., the very fact of the question is public, (2) only the agent i knows the answer, i.e., the answer is private, (3) the question requires the agent i to spend some amount of resources. All of these features will be described formally in this section.

We extend the truth relation $\vDash$ introduced in Definition 3 to the dynamic operator $[?_i A]\varphi$ as follows.

Definition 5. Given a model $\mathcal{M} = (W, (\sim_i)_{i \in \mathsf{Agt}}, \mathsf{Cost}, \mathsf{Bdg}, V)$ and a state $w \in W$,

$$\mathcal{M}, w \vDash [?_i A]\varphi \quad \text{iff} \quad \mathcal{M}, w \vDash (b_i \geq c_A) \text{ implies } \mathcal{M}^{?_i A}, w \vDash \varphi.$$

Here $\mathcal{M}^{?_i A}$ is a model obtained from $\mathcal{M}$ by the update that corresponds to the following action: "*the agent i asked whether the propositional formula A is true and spent for this the amount of resources* $\mathsf{Cost}(A)$"; the updated model is described in the next definition. We will use notation: $[A]_{\mathcal{M}} := \{w \in W \mid \mathcal{M}, w \vDash A\}$.

Definition 6. Given a model $\mathcal{M} = (W, (\sim_i)_{i \in \mathsf{Agt}}, \mathsf{Cost}, \mathsf{Bdg}, V)$, an *updated model* is a tuple $\mathcal{M}^{?_i A} = (W^{?_i A}, (\sim_j^{?_i A})_{j \in \mathsf{Agt}}, \mathsf{Cost}^{?_i A}, \mathsf{Bdg}^{?_i A}, V^{?_i A})$, where

- $W^{?_i A} = \{w \in W \mid \mathcal{M}, w \models b_i \geq c_A\}$,
- $\sim_j^{?_i A} = (W^{?_i A} \times W^{?_i A}) \cap \sim_j^*$,
  where $\sim_j^* = \begin{cases} \sim_j \bigcap \Big(([A]_{\mathcal{M}} \times [A]_{\mathcal{M}}) \bigcup ([\neg A]_{\mathcal{M}} \times [\neg A]_{\mathcal{M}}) \Big) & \text{if } j = i, \\ \sim_j & \text{otherwise,} \end{cases}$
- $\mathsf{Cost}^{?_i A}(B) = \mathsf{Cost}(B)$, for all propositional formulas B,
- $\mathsf{Bdg}_j^{?_i A}(w) = \begin{cases} \mathsf{Bdg}_j(w) - \mathsf{Cost}(A, w), & \text{if } j = i, \\ \mathsf{Bdg}_j(w), & \text{otherwise,} \end{cases}$
- $V^{?_i A}(p) = V(p) \cap W^{?_i A}$.

Intuitively, the update $[?_i A]\varphi$ of model $\mathcal{M}$ firstly removes all states of $\mathcal{M}$ in which agent i does not have a sufficient amount of resources to ask about A. This can be justified by the fact that other agents do not necessarily know i's budget, but when they observe the fact that i actually asks about the truth of A, it no longer makes sense to consider the states with $(b_i < c_A)$ as possible ones. Secondly, when i asks *"is A true?"*, she gets either *"Yes"* or *"No"* and we consider this fact to be known by all agents. Then, after this update, the agent i necessarily distinguishes any two states of $\mathcal{M}$ that do not agree on the valuation of A. But since the actual answer is available only to the agent i, the epistemic relations of other agents remain the same, only taking into account that some states have been removed. This update does not affect the costs of formulas and budgets of all agents except i. Budget of i decreases by the cost of A after $[?_i A]$. As one can see, all of these assumptions sound quite natural.

Consider an example with two agents i and j. Let p_i stand for 'i is COVID-positive' and p_j stands for 'j is COVID-positive'. Assume that the cost of the test is 20 resources in all possible worlds ($\mathcal{M} \vDash c_{p_i} = 20 \wedge c_{p_j} = 20$). If we also assume that i decides to make the test ($[?_i p_i]$), then the semantics of $\mathsf{DEL_{bc}}$ describes this situation as presented in Figure 1.

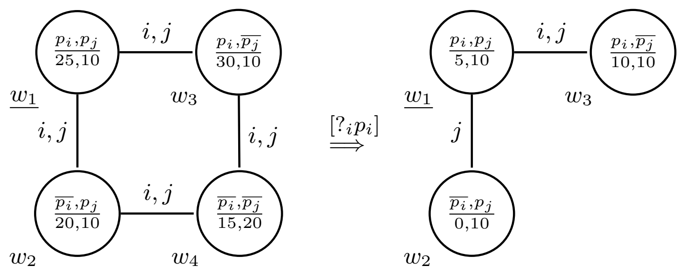

Fig. 1. Initial model $\mathcal{M}$ and updated model $\mathcal{M}^{?_i p_i}$

Note that an agent does not necessarily knows even her own budget. The following formulas hold in w_1:

- $\mathcal{M}, w_1 \vDash \neg K_i p_i$
- $\mathcal{M}^{?_i p_i}, w_1 \vDash K_i p_i$
- $\mathcal{M}^{?_i p_i}, w_1 \vDash \neg K_j p_i$
- $\mathcal{M}^{?_i p_i}, w_1 \vDash K_j K_i^? p_i$
- $\mathcal{M}, w_1 \vDash \neg K_i (b_i \geq 20)$
- $\mathcal{M}^{?_i p_i}, w_1 \vDash K_i (b_i \geq 0)$
- $\mathcal{M}, w_1 \vDash \neg K_j (b_j = 10)$
- $\mathcal{M}^{?_i p_i}, w_1 \vDash K_j (b_j = 10)$

3.3 Some Valid Formulas

Here we present some examples of valid formulas w.r.t. the proposed semantics.

Proposition 1. $\vDash (b_i \geq c_A) \leftrightarrow \langle ?_i A \rangle \top$.

Proof. $\mathcal{M}, w \vDash \langle ?_i A \rangle \top$ is equivalent to $\mathcal{M}, w \vDash \neg [?_i A] \bot$ by definition of $\langle ?_i A \rangle$. Then $\mathcal{M}, w \vDash \neg [?_i A] \bot$ is equivalent to $\mathcal{M}, w \vDash (b_i \geq c_A)$ and $\mathcal{M}^{?_i A}, w \vDash \top$. But since $w \in W^{?_i A}$ iff $\mathcal{M}, w \vDash (b_i \geq c_A)$, then $\mathcal{M}^{?_i A}, w \vDash \top$ is also equivalent to $\mathcal{M}, w \vDash (b_i \geq c_A)$.

Proposition 2. $\vDash \langle ?_i A \rangle \varphi \rightarrow [?_i A] \varphi$.

Proof. As we mentioned above, $\mathcal{M}, w \vDash \langle ?_i A \rangle \varphi$ is equivalent to $\mathcal{M}, w \vDash (b_i \geq c_A)$ and $\mathcal{M}^{?_i A}, w \vDash \varphi$. This conjunction obviously implies that $\mathcal{M}, w \vDash (b_i \geq c_A) \Rightarrow \mathcal{M}^{?_i A}, w \vDash \varphi$.

Proposition 3. $\vDash [?_i A] K_i^? A$.

Proof. It is clear that $w \sim^{?_i A} w'$ implies ($\mathcal{M}, w \vDash A$ and $\mathcal{M}, w' \vDash A$) or ($\mathcal{M}, w \vDash \neg A$ and $\mathcal{M}, w' \vDash \neg A$) by Definition 6. Then $\mathcal{M}, w \vDash (b_i \geq c_A)$ implies $\mathcal{M}^{?_i A}, w \vDash (K_i A \vee K_i \neg A)$.

3.4 Soundness and Completeness

Axiomatization of $\mathsf{DEL_{bc}}$ can be obtained by adding the reduction axioms from Table 3 to the axiomatization of $\mathsf{EL_{bc}}$. The notation $\big((z_1 t_1 + \cdots + z_n t_n) \geq z)\big)^{[b_i \setminus (b_i - c_A)]}$ means that all occurrences of b_i in $(z_1 t_1 + \cdots + z_n t_n) \geq z$ are replaced with $(b_i - c_A)$.

Table 3. Reduction axioms and inference rules

(R_p)	$[?_i A]p \leftrightarrow (b_i \geq c_A) \to p$
$(R_{\geq})$	$[?_i A]\big((z_1 t_1 + \cdots + z_n t_n) \geq z)\big) \leftrightarrow (b_i \geq c_A) \to$ $\to \big((z_1 t_1 + \cdots + z_n t_n) \geq z)\big)^{[b_i \setminus (b_i - c_A)]}$
$(R_{\neg})$	$[?_i A]\neg\varphi \leftrightarrow (b_i \geq c_A) \to \neg[?_i A]\varphi$
$(R_{\wedge})$	$[?_i A](\varphi \wedge \psi) \leftrightarrow [?_i A]\varphi \wedge [?_i A]\psi$
(R_{K_j})	$[?_i A]K_j \varphi \leftrightarrow (b_i \geq c_A) \to K_j[?_i A]\varphi$, where $i \neq j$
(R_{K_i})	$[?_i A]K_i \varphi \leftrightarrow (b_i \geq c_A) \to$ $\to \Big(\big(A \to K_i(A \to [?_i A]\varphi)\big) \wedge \big(\neg A \to K_i(\neg A \to [?_i A]\varphi)\big)\Big)$
(Rep)	From $\vdash \varphi \leftrightarrow \psi$, infer $\vdash [?_i A]\varphi \leftrightarrow [?_i A]\psi$

Proposition 4. *Axioms (R_p), $(R_{\neg})$, and $(R_{\wedge})$ and inference rule Rep are sound w.r.t. $\mathfrak{M}$.*

Proof. Trivial.

Lemma 2. *For $i \neq j$ is holds that $w \sim_j^{?_i A} w'$ iff $w \sim_j w'$, $\mathcal{M}, w \vDash (b_i \geq c_A)$, and $\mathcal{M}, w \vDash (b_i \geq c_A)$.*

Proof. Follows straightforward from Definition 6.

Proposition 5. *For any model $\mathcal{M}$ and any point $w \in W$, it holds that*

$$\mathcal{M}, w \vDash [?_i A]K_j \varphi \quad \textit{iff} \quad \mathcal{M}, w \vDash (b_i \geq c_A) \to K_j[?_i A]\varphi, \textit{ where } i \neq j.$$

Proof. ($\Rightarrow$) Let $\mathcal{M}, w \vDash [?_i A]K_j \varphi$ (1) and $\mathcal{M}, w \vDash (b_i \geq c_A)$ (2). From (1), $\mathcal{M}, w \vDash (b_i \geq c_A)$ implies $\mathcal{M}^{?_i A}, w \vDash K_j \varphi$ (1.1) by Definition 5. Then $\mathcal{M}^{?_i A}, w \vDash K_j \varphi$ from (1.1) and (2). Then $\forall w' : (w \sim_j^{?_i A} w') \Rightarrow \mathcal{M}^{?_i A}, w' \vDash \varphi$ by Definition 3. This fact together with Lemma 2 implies that $\forall w'(w \sim_j w')$: $\mathcal{M}, w' \vDash (b_i \geq c_A) \Rightarrow \mathcal{M}^{?_i A}, w' \vDash \varphi$. This is equivalent to $\mathcal{M}, w \vDash K_j[?_i A]\varphi$, by Definition 3 and Definition 5.

($\Leftarrow$) The case for $\mathcal{M}, w \nvDash (b_i \geq c_A)$ is trivial. Consider only the case for $\mathcal{M}, w \vDash K_j[?_i A]\varphi$. Then $\forall w'(w \sim_j w') : \mathcal{M}, w' \vDash (b_i \geq c_A) \Rightarrow \mathcal{M}^{?_i A}, w' \vDash \varphi$. By Lemma 2 it holds that $\forall w' : w \sim_j^{?_i A} w' \Rightarrow \mathcal{M}^{?_i A}, w' \vDash \varphi$. By Definition 3, $\mathcal{M}^{?_i A}, w \vDash K_j\varphi$ and hence $\mathcal{M}, w \vDash [?_i A]K_j\varphi$.

Lemma 3.

- *$w \sim_i^{?_i A} w'$ iff $w \sim_i w'$ (1), $\mathcal{M}, w \vDash (b_i \geq c_A)$ (2.1), $\mathcal{M}, w' \vDash (b_i \geq c_A)$ (2.2) and $w \approx_A w'$ (3), where $w \approx_A w'$ holds if either both $\mathcal{M}, w \vDash A$ and $\mathcal{M}, w' \vDash A$ hold or both $\mathcal{M}, w \vDash \neg A$ and $\mathcal{M}, w' \vDash \neg A$ hold,*
- *$\mathcal{M}, w \vDash A$ iff $\mathcal{M}^{?_i A}, w \vDash A$, where A is a propositional formula.*

Proof. Follows straightforwardly from Definition 6.

Proposition 6. *For any model $\mathcal{M}$ and any point $w \in W$, we have:*

$$\mathcal{M}, w \vDash [?_i A]K_i\varphi \text{ iff } \mathcal{M}, w \vDash (b_i \geq c_A) \to \bigwedge_{A' \in \{A, \neg A\}} \big(A' \to K_i(A' \to [?_i A']\varphi)\big).$$

Proof. ($\Rightarrow$) Let $\mathcal{M}, w \vDash [?_i A]K_i\varphi$ (1) and $\mathcal{M}, w \vDash (b_i \geq c_A)$ (2). From (1), (2) and Definition 5 we get $\mathcal{M}^{?_i A}, w \vDash K_i\varphi$. Then $\forall w'(w \sim_i^{?_i A} w') \Rightarrow \mathcal{M}^{?_i A}, w' \vDash \varphi$. Assume that $\mathcal{M}, w \vDash A$. Then by Lemma 3 it follows that $\forall w' : w \sim_i w'$ and $\mathcal{M}, w' \vDash A$ and $\mathcal{M}, w' \vDash (b_i \geq c_A)$ implies $\mathcal{M}^{?_i A}, w' \vDash \varphi$. This is equivalent to $\mathcal{M}, w \vDash K_i(A \to [?_i A]\varphi)$ by Definition 3 and Definition 5. Then, from our assumption we proved that $\mathcal{M}, w \vDash A \to K_i(A \to [?_i A]\varphi)$. By a similar argument, one can show that $\mathcal{M}, w \vDash \neg A \to K_i(\neg A \to [?_i A]\varphi)$.

($\Leftarrow$) The case for $\mathcal{M}, w \nvDash (b_i \geq c_A)$ is trivial. Consider only the case for $\mathcal{M}, w \vDash \bigwedge_{A' \in \{A, \neg A\}} \big(A' \to K_i(A' \to [?_i A']\varphi)\big)$. Assume that $\mathcal{M}, w \vDash A$. Then $\mathcal{M}, w \vDash K_i(A \to [?_i A]\varphi)$. Similarly, assuming $\mathcal{M}, w \vDash \neg A$ entails $\mathcal{M}, w \vDash K_i(\neg A \to [?_i A]\varphi)$. Then for all w', such that $(w \sim_i w')$ and w' agrees with w on the valuation of A it holds that $\mathcal{M}, w' \vDash [?_i A]\varphi$ and hence $\mathcal{M}, w' \vDash (b_i \geq c_A)$ implies $\mathcal{M}^{?_i A}, w' \vDash \varphi$. Then by Lemma 3 it holds that $\forall w' : w \sim_i^{?_i A} w' \Rightarrow \mathcal{M}^{?_i A}, w' \vDash \varphi$. And hence $\mathcal{M}^{?_i A}, w \vDash K_i\varphi$. By Definition 5, the last claim implies $\mathcal{M}, w \vDash [?_i A]K_i\varphi$.

Proposition 7. *Axiom $(R_{\geq})$ is sound w.r.t. $\mathfrak{M}$*

Proof. It is clear that $\mathcal{M}, w \vDash [?_i A](z_1t_1 + \cdots + z_nt_n) \geq z$ iff $\mathcal{M}, w \vDash (b_i \geq c_A)$ implies $\mathcal{M}^{?_i A}, w \vDash (z_1t_1 + \cdots + z_nt_n) \geq z$ by Definition 5. Note that $\mathcal{M}^{?_i A}, w \vDash (z_1t_1 + \cdots + z_nt_n) \geq z$ is equivalent to $\mathcal{M}, w \vDash (z_1t_1^* + \cdots + z_nt_n^*) \geq z$, where $t_k^* = t_k$ for $t_k = c_A$ or $t_k = b_j$. And $t_k^* = t_k + \mathsf{Cost}(A)$ for $t_k = b_i$ since $\mathsf{Cost}^{?_i A}(B) = \mathsf{Cost}(B)$, $\mathsf{Bdg}_j^{?_i A}(w) = \mathsf{Bdg}_j(w)$ for $i \neq j$ and $\mathsf{Bdg}_i^{?_i A}(w) = \mathsf{Bdg}_i(w) - \mathsf{Cost}(A)$. Then $\mathcal{M}, w \vDash [?_i A](z_1t_1 + \cdots + z_nt_n) \geq z$ iff $\mathcal{M}, w \vDash (b_i \geq c_A)$ implies $\mathcal{M}, w \vDash [(z_1t_1 + \cdots + z_nt_n) \geq z)]^{[b_i \setminus (b_i - c_A)]}$.

Theorem 5 (Soundness). $\mathsf{DEL_{bc}}$ *is sound w.r.t. $\mathfrak{M}$, i.e. $\vdash_{\mathsf{DEL_{bc}}} \varphi \Longrightarrow \vDash_{\mathfrak{M}} \varphi$*

Proof. Follows from Proposition 4 – Proposition 7.

Theorem 6 (Completeness). $\mathsf{DEL_{bc}}$ *is complete w.r.t.* $\mathfrak{M}$, *i.e.* $\vdash_{\mathsf{DEL_{bc}}} \varphi$ *iff* $\models_{\mathfrak{M}} \varphi$

Proof. Left-to-right direction follows from Theorem 5. The other direction holds by Theorem 2 and the standard for dynamic epistemic logic completeness via reduction argument.

Theorem 7 (Decidability). *The satisfiability problem for* $\mathsf{DEL_{bc}}$ *is decidable.*

Proof. This result is straightforward since any $\mathsf{DEL_{bc}}$ formula can be translated into $\mathsf{EL_{bc}}$ formula in finitely many steps by the rules presented in Table 3 and the decidability of $\mathsf{EL_{bc}}$ is demonstrated in Theorem 4.

4 Combination of $\mathsf{DEL_{bc}}$ and PAL

The language $\mathsf{DEL_{bc!}}$ extends the language $\mathsf{DEL_{bc}}$ with a standard operator for public announcement $[!\varphi]$. A formula $[!\varphi]\psi$ stands for "after public announcement of φ, it holds that ψ ".

Definition 7. The *formulas* of $\mathsf{DEL_{bc!}}$ are defined by the following grammar:

$$\varphi, \psi ::= p \mid (z_1 t_1 + \cdots + z_n t_n) \geq z) \mid \neg\varphi \mid (\varphi \wedge \psi) \mid K_i\varphi \mid [?_i A]\varphi, \mid [!\varphi]\psi$$

where $p \in \mathsf{Prop}$, $A \in \mathcal{L}_{PL}$, $i \in Agt$, $t_1, \ldots, t_n \in \mathsf{Const}$ and $z_1, \ldots, z_n, z \in \mathbb{Z}$.

Definition 8. $\mathcal{M}, w \models [!\varphi]\psi \iff \mathcal{M}, w \models \varphi \Rightarrow \mathcal{M}^{!\varphi}, w \models \psi$, *where* $\mathcal{M}$ *is defined in Definition 2 and* $\mathcal{M}^{!\varphi}$ *is a model* $\mathcal{M}$ *restricted to* φ*-worlds.*

Rational question We will call the question rational if the agent doesn't know the answer to this question. We can express the condition for a rational question in $\mathsf{DEL_{bc!}}$ as $[?^r_i A]\varphi := [!\neg K^?_i A][?_i A]\varphi$. A formula $[?^r_i A]\varphi$ can be read as "φ is true after i's rational question whether A is true".

Example [3 cards puzzle] From a pack of three known cards X, Y, Z, Alice, Bob and Cath each draw one card. Initially, all agents has zero points. If an agent has X or Y, then its score increases by one point. Also, from a pack of three known card $1, 0, 0$ each agent draws one card. If an agent has 1, then its score increases by one point, 0 does not change anything. An agent may ask a question publicly and get an answer (yes or no) privately. The cost of any question is 1 point. Bob asks: "Whether Cath has the card Y?". Alice says "I know that my points and Bob's points are different". Cath says "I know the cards".

We can represent the initial situation with a Figure 2. The sequence of updates can be formalized as follows:

$$\langle ?^r_b Y_c\rangle\langle !K_a(b_a \neq b_b)\rangle\langle !K_c(XYZ)_?\rangle\top$$

Here $K_i(XYZ)_? := K^?_i X_? \wedge K^?_i Y_? \wedge K^?_i Z_?$ and $K^?_i X_? := K^?_i X_a \wedge K^?_i X_b \wedge K^?_i X_c$ (similarly for Y and Z). The results of updates are presented in Figure 3. Hence, the only one possible world satisfies this series of updates.

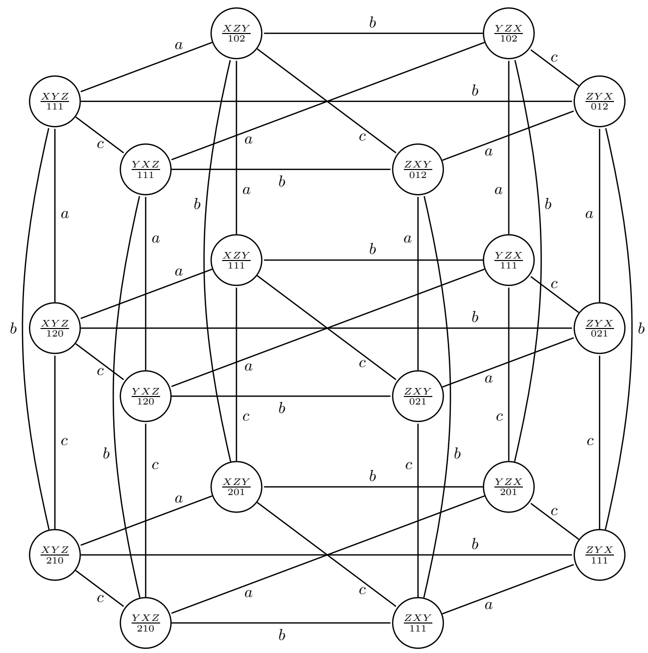

Fig. 2. Model for "3 cards" puzzle. Reflexivity, symmetry and transitivity are assumed.

Axiomatisation The sound and complete axiomatisation for $\mathsf{DEL}_{\mathsf{bc!}}$ can be obtained as a combination of $\mathsf{DEL}_{\mathsf{bc}}$ and PAL (see [18]) proof systems with an additional reduction axiom: $[!\varphi]((z_1t_1 + \cdots + z_nt_n) \geq z) \leftrightarrow (\varphi \rightarrow (z_1t_1 + \cdots + z_nt_n) \geq z)$

5 Discussion

In this paper, we present $\mathsf{EL}_{\mathsf{bc}}$, a static epistemic logic for budget-constrained agents, and provide its sound and complete axiomatisation. Then we present $\mathsf{DEL}_{\mathsf{bc}}$, a dynamic epistemic logic for budget-constrained agents, which extends $\mathsf{EL}_{\mathsf{bc}}$ with dynamic operator $[?_iA]\varphi$. For the dynamic fragment, we provide sound reduction axioms demonstrating $\mathsf{DEL}_{\mathsf{bc}}$ completeness via a reduction argument. The proposed logics are sufficiently expressive to deal with non-trivial epistemic scenarios involving reasoning about costs of propositional formulas and agents' budgets. In addition, $\mathsf{DEL}_{\mathsf{bc}}$ is able to describe the semantics of a special class of questions. These questions can be asked publicly, but the answer is available only to the asking agent. Moreover, to get an answer, an agent must spend some resources, thus decreasing her budget. This gives rise to a new direction of research in the field of reasoning about resource-bounded agents in multi-agent systems allowing to formalise not only inner or cognitive resources, but also external resources as obstacles in the process of obtaining new information from the environment.

It is worth noting that we make some assumptions about the properties of Cost and Bdg functions. Firstly, we assume that costs of formulas depend on a particular state of

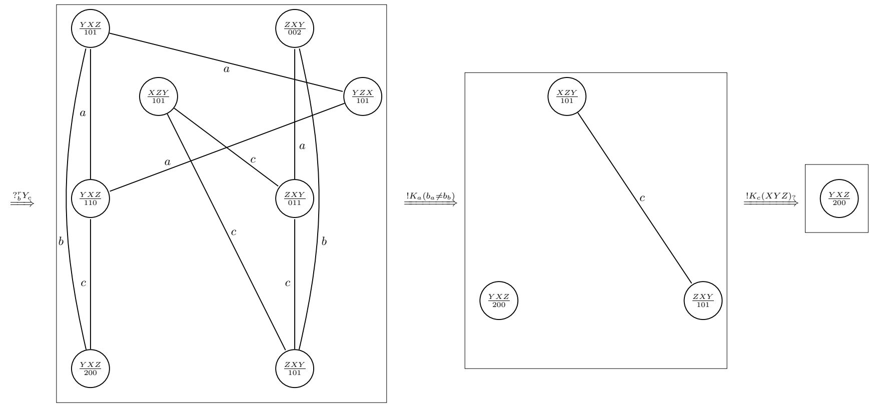

Fig. 3. Models for "3 cards" puzzle in a series of updates. Reflexivity, symmetry and transitivity are assumed.

a model, i.e. some formula can have different costs in different states. This assumption allows us to model situations in which an agent does not necessarily know the cost of some formula. Our second assumption is that agents do not necessarily know the budget of other agents as well as their own. But this assumption can be eased by introducing additional axioms as we demonstrate in Theorem 6. Our last assumption deals with the relationship between the costs of different formulas. The fact that equivalent formulas must have equal costs seems obvious. It is also plausible that $\mathsf{Cost}(A)$ must be equal to $\mathsf{Cost}(\neg A)$, since asking questions "Is A true?" and "Is $\neg A$ true?" can be considered as the same informational action. But these are the only constraints on the Cost function we imposed in this paper. It remains an open question how to deal with Boolean connectives in the sense of their costs. As a future work, one of our aims is to deal with this aspect. For example, it looks quite natural to consider the following property: $c_A + c_B \geq c_{A \circ B}$, where $\circ$ is any Boolean connective.

As for the $\mathsf{DEL_{bc}}$ extension, it is natural to introduce additional dynamic modalities: an operator $[?_G A]\psi$ which involves sharing resources among a group of agents, G and an operator $\langle ?_i^n \rangle \varphi$ for existential quantification over updates (there is a propositional formula, A, such that the cost of A is at most, n, and it is true that $\langle ?_i A \rangle \varphi$). This would allow us to define a concept such as n-knowability, meaning that φ is knowable given n resources. Finally, in future work, we plan to establish complexity results for the satisfiability problem and investigate model-checking algorithms for our logics.

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References

1. Alechina, N., Logan B.: Ascribing Beliefs to Resource Bounded Agents. In: Proceedings of the First International Joint Conference on Autonomous Agents and Multiagent Systems, pp. 881–888. Association for Computing Machinery, New York (2002)

2. Alechina N., Logan B.: A Logic of Situated Resource-Bounded Agents. Journal of Logic, Language and Information **18**, 79–95 (2009)
3. Alechina N., Logan B., Whitsey M.: A complete and decidable logic for resource-bounded agents. In: Proc. AAMAS, 606–613, New York, USA, (2004)
4. Balbiani P., Fernandez-Duque D., Lorini E.: The Dynamics of Epistemic Attitudes in Resource-Bounded Agents. Studia Logica **107**, 457–488 (2019)
5. van Ditmarsch H., Fan J.: Propositional quantification in logics of contingency. Journal of Applied Non-Classical Logics **26**(1), 81–102 (2016)
6. van Ditmarsch, H.P., van der Hoek, W., Kooi, B.: Dynamic Epistemic Logic. Springer, Berlin (2007)
7. Dautović, S., Doder, D., Ognjanović, Z.: An epistemic probabilistic logic with conditional probabilities, in: W. Faber, G. Friedrich, M. Gebser, M. Morak (Eds.), Logics in Artificial Intelligence, Springer International Publishing, Cham, pp. 279–293 (2021)
8. Duc, H. N.: Reasoning about rational, but not logically omniscient, agents, Journal of Logic and Computation 7(5):633–648 (1997)
9. Elgot-Drapkin, J., Miller, M., and Perlis, D.: Memory, reason and time: the steplogic approach, in R. Cummins and J. L. Pollock, (eds.), Philosophy and AI: Essays at the Interface, MIT Press (1991)
10. Fagin R., Halpern J.Y.: Belief, Awareness, and Limited Reasoning. Artificial Intelligence **34**, 39–76 (1988)
11. Fagin R., Halpern J. Y., Megiddo N.: A logic for reasoning about probabilities. Information and Computation, **87**(1), 78–128 (1990)
12. Fagin R., Halpern J.Y., Moses Y., and Vardi M.Y.: Reasoning about knowledge. MIT Press, Cambridge, MA (1995)
13. Grant, J., Kraus S., and Perlis, D.: A logic for characterizing multiple bounded agents, Autonomous Agents and Multi-Agent Systems 3(4):351–387 (2000)
14. Jago, M.: Epistemic Logic for Rule-Based Agents, Journal of Logic, Language and Information 18(1):131–158 (2009)
15. Kooi, B.: Dynamic Epistemic Logic. Handbook of Logic and Language. 2nd edn. 671–690 (2011)
16. Naumov P., Tao J.: Budget-constrained Knowledge in Multiagent Systems. In: Proc. AAMAS, 219–226, Istanbul (2015)
17. Solaki A.: Bounded Multi-agent Reasoning: Actualizing Distributed Knowledge. In: DaLi 2020: Dynamic Logic. New Trends and Applications 2020, LNCS, volume 12569 pp. 239–258 (2020)
18. Wang Y., Cao, Q.: On Axiomatizations of Public Announcement Logic, Synthese, 190: 103–134 (2013)

Action Models for Coalition Logic

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Abstract. In the paper, we study the dynamics of coalitional ability by proposing an extension of coalition logic (CL). CL allows one to reason about what a coalition of agents is able to achieve through a joint action, no matter what agents outside of the coalition do. The proposed dynamic extension is inspired by dynamic epistemic logic, and, in particular, by action models. We call the resulting logic coalition action model logic (CAML), which, compared to CL, includes additional modalities for coalitional action models. We investigate the expressivity of CAML, and provide a complexity characterisation of its model checking problem.

Keywords: Dynamic Coalition Logic · Coalition Logic · Dynamic Epistemic Logic · Action Model Logic.

1 Introduction

Coalition logic (CL) [18, 17] is one of the most well-known formalisms for reasoning about strategic abilities of groups of agents in the presence of opponents. Modalities $\langle\!\langle C \rangle\!\rangle \varphi$ of CL express the fact that 'there is a joint action by agents from coalition C such that no matter what agents outside of the coalition do at the same time, φ will be true'. CL was conceived as a formal language for strategic games, and constructs $\langle\!\langle C \rangle\!\rangle \varphi$ characterise the existence of a winning strategy for agents in C.

One way to approach models of CL is to view them as protocols or contracts specifying what agents can and cannot do in different states. In this paper, we propose an extension of CL, which we call coalition action model logic (CAML), that includes modalities for updating those models. Such updates are carried out with respect to *action models* that are expressed in the language with formulas $[\mathsf{M_s}]\varphi$ meaning 'after executing action model $\mathsf{M_s}$, φ is the case'.

In creating CAML we followed the lead of dynamic epistemic logic (DEL) [11], and, in particular, of action model logic (AML) [9, 11]. Action models in AML model various epistemic events that can influence agents' knowledge about facts of the world and about knowledge of other agents. In a similar vein, coalitional action models of CAML can influence strategic abilities of coalitions of agents. On a more general scale, we hope that CAML will be a step towards a study of *dynamic coalition logic* (DCL). To make the link between DEL and DCL even more explicit, we can say that while DEL captures the *the dynamics of knowledge*, DCL should be able to capture the *dynamics of ability*.

CAML is not the first dynamic coalition logic. In [12] the authors proposed dictatorial dynamic coalition logic (DDCL) that was inspired by arrow update logic [15] and relation-changing logics [6]. DDCL updates strategic abilities of single agents by granting them dictatorial powers or revoking such powers. Compared to DDCL, action models of CAML allow for more fine-tuned updates that may affect more than one agent in various ways. On the other hand, modalities of CAML neither grant agents new actions they have not had before, nor remove such actions. Thus, coalitional action models can be viewed as prescriptions of how protocols or contracts between agents should be modified while taking into account what agents actually can and cannot do.

The implementation of an action model in CAML is not, however, merely a restriction (submodel) of the initial model. The action model might prescribe several different modifications compatible with the same state in the initial model, and the resulting updated model might have *more* states than the initial one. We capture this by using a definition of a product update very similar to the one used in AML. Restrictions on transition systems corresponding to policies, norms, or social laws is far from a new idea [20, 21], and logical formalisms for reasoning about such restrictions have been extensively studied, particularly using systems based on computation tree logic (CTL) [3]. In [4] a language similar to CL is used: an expression of the form $\langle C \rangle \varphi$, where C is a coalition and φ is a temporal formula, expresses the fact that if coalition C *complies* with the normative system, then φ will be true. Here, formulas are interpreted in the context of a single, given, restriction on legal actions, and although one can quantify over different parts of that restriction by varying the coalition C, the resulting submodel will always be a *restriction* of the initial model. The conceptual overlap notwithstanding, CAML is significantly different: as mentioned earlier, the updated models, obtained using action models, are not necessarily submodels, and the underlying models are CL models with joint actions rather than Kripke models of CTL.

After we briefly present necessary background information on CL in Section 2, we introduce CAML in Section 3. In Section 4 we show that CAML is strictly more expressive than CL. This result shows a crucial difference between AML and CAML: whereas the former is as expressive as the underlying epistemic logic, and thus completeness of AML follows trivially from reduction axioms, we cannot have reduction axioms for CAML. Moreover, we claim that CAML is incomparable to alternating-time temporal logic (ATL) [5]. Added expressivity of CAML comes at a price. In Section 5 we show that the complexity of the model checking problem jumps from P for CL to $PSPACE$-complete for CAML. Finally, we conclude in Section 6.

2 Language and Semantics of Coalition Logic

In this section we briefly provide all the necessary background information on coalition logic [2, 17]. Let A be a finite set of agents, and P be a countable set of propositional variables.

Definition 1. *The* language of coalition logic $\mathcal{CL}$ *is defined by BNF:*

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \langle\!\langle C \rangle\!\rangle \varphi$$

where $p \in P$ *and* $C \subseteq A$. *Formulas* $\langle\!\langle C \rangle\!\rangle \varphi$ *are read 'coalition* C *can force* φ'. *We denote* $A \setminus C$ *as* $\overline{C}$. *The dual of* $\langle\!\langle C \rangle\!\rangle \varphi$ *is* $[\![C]\!]\varphi := \neg\langle\!\langle C \rangle\!\rangle \neg\varphi$. *We will call subsets of* A *'coalitions', and we will also call complements of* C, $\overline{C}$, *'the anti-coalition'.*

The semantics of CL is given with respect to concurrent game models. A *concurrent game model* (CGM), or a *model*, is a tuple $M = (S, Act, act, out, L)$. S is a non-empty set of states, and Act is a non-empty set of actions.

The function $act : A \times S \to 2^{Act} \setminus \emptyset$ assigns to each agent and each state a non-empty set of actions. A C-action at a state $s \in S$ is a tuple α_C such that $\alpha_C(i) \in act(i, s)$ for all $i \in C$. The set of all C-actions in s is denoted by $act(C, s)$. We will also write $\alpha_{C_1} \cup \alpha_{C_2}$ to denote a $C_1 \cup C_2$-action with $C_1 \cap C_2 = \emptyset$.

A tuple of actions $\alpha = \langle \alpha_1, \ldots, \alpha_k \rangle$ with $k = \mid A \mid$ is called an *action profile*. An action profile is *executable* in state s if for all $i \in A$, $\alpha_i \in act(i, s)$. The set of all action profiles executable in s is denoted by $act(s)$. An action profile α *extends* a C-action α_C, written $\alpha_C \sqsubseteq \alpha$, if for all $i \in C$, $\alpha(i) = \alpha_C(i)$.

The function out assigns to each state s and each $\alpha \in act(s)$ a unique output state. We write $Out(s, \alpha_C)$ for $\{out(s, \alpha) \mid \alpha \in act(s) \text{ and } \alpha_C \sqsubseteq \alpha\}$. Intuitively, $Out(s, \alpha_C)$ is the set of all states reachable by action profiles that extend some given C-action α_C. Finally, $L : S \to 2^P$ is the valuation function.

We will also denote a CGM M with a designated, or current, state s as M_s, and will sometimes call it a *pointed model*. We call M *finite* if S is finite.

Definition 2. *Let* M_s *be a pointed CGM. The semantics of CL is defined inductively as follows:*

$$\begin{array}{ll}
M_s \models p & \textit{iff } s \in L(p) \\
M_s \models \neg\varphi & \textit{iff } M_s \not\models \varphi \\
M_s \models \varphi \wedge \psi & \textit{iff } M_s \models \varphi \textit{ and } M_s \models \psi \\
M_s \models \langle\!\langle C \rangle\!\rangle \varphi & \textit{iff } \exists \alpha_C, \forall \alpha_{\overline{C}} : M_t \models \varphi, \textit{ where } t = out(s, \alpha_C \cup \alpha_{\overline{C}}) \\
M_s \models [\![C]\!]\varphi & \textit{iff } \forall \alpha_C, \exists \alpha_{\overline{C}} : M_t \models \varphi, \textit{ where } t = out(s, \alpha_C \cup \alpha_{\overline{C}})
\end{array}$$

Informally, the semantics of the coalition modality $\langle\!\langle C \rangle\!\rangle \varphi$ means that in the current state of a given CGM there is a choice of actions by the members of coalition C such that no matter what the opponents from the anti-coalition $\overline{C}$ choose to do at the same time, φ holds after the execution of the corresponding action profile.

Definition 3. *We call a formula* φ valid *if for all* M_s *it holds that* $M_s \models \varphi$.

Example 1. An example of a CGM is presented in Figure 1 on the left. The model is called M and it describes the following protocol. There are two states: s, where agents receive a *p*rize (propositional variable p), and state t, where agents do not receive a prize. Each agent has two actions in each state, and they can switch states by 'synchronisation', i.e. by choosing actions with the same number, either $a_0 b_0$ or $a_1 b_1$. Formally, $M_s \models p \wedge \langle\!\langle \{a, b\} \rangle\!\rangle \neg p$. At the same time, no agent alone can force the transition to state t, or, in symbols, $M_s \models [\![a]\!]p \wedge [\![b]\!]p$.

The classic notion of indistinguishability between models in modal logic is bisimulation. In this paper, we will use a CGM-specific version of bisimulation [1].

Definition 4. *Let $M = (S^M, Act^M, act^M, out^M, L^M)$ and $N = (S^N, Act^N, act^N, out^N, L^N)$ be two CGMs. A relation $Z \subseteq S^M \times S^N$ is called* bisimulation *if and only if for all $C \subseteq A$, $s_1 \in S^M$ and $s_2 \in S^N$, $(s_1, s_2) \in Z$ implies*

- *for all $p \in P$, $s_1 \in L^M(p)$ iff $s_2 \in L^N(p)$;*
- *for all $\alpha_C \in act^M(C, s_1)$, there exists $\beta_C \in act^N(C, s_2)$ such that for every $s_2' \in Out^N(s_2, \beta_C)$, there exists $s_1' \in Out^M(s_1, \alpha_C)$ such that $(s_1', s_2') \in Z$.*
- *The same as above with 1 and 2 swapped.*

If there is a bisimulation between M and N linking states s_1 and s_2, we call the pointed models bisimilar *($M_{s_1} \leftrightarrows N_{s_2}$).*

The crucial property of bisimilar models, that will be of use later in the paper, is that bisimilar models satisfy the same set of formulas of coalition logic.

Theorem 1 ([1]). *Let M and N be CGMs such that $M \leftrightarrows N$ and there is a bisimulation between $s \in S^M$ and $t \in S^N$. Then for all $\varphi \in \mathcal{CL}$, $M_s \models \varphi$ iff $N_t \models \varphi$.*

3 Coalition Action Model Logic

Before providing formal definitions, we introduce coalitional action models intuitively with an example.

3.1 Informal Exposition and Example

Coalitional action models are inspired by action models of DEL [9, 11], and they are, basically, models, where each state has an assigned formula that is called a *precondition*. Preconditions indicate which states of action models are executable in which states of a given CGM. An action model can be viewed as a *policy*, explicitly describing *legal* joint actions and implicitly imposing *restrictions* on existing joint actions in different states. However, the result of implementing an action model policy is not necessarily a submodel of the initial model: it can in fact have more states, if the action model describes more possible actions compatible with the same state in the initial model. We capture this by a *product update* using a restricted Cartesian product, very similar to product updates in AML. In particular, in the update of a CGM with an action model, we take a product of states of the CGM and those states of the action model that are satisfied according to preconditions. In the resulting *updated model*, a transition labelled with an action profile is preserved, if there is a corresponding transition in both CGM and the action model. To differentiate CGMs and coalitional action models, we will use $\mathsf{sans\text{-}serif}$ font for the elements of the latter.

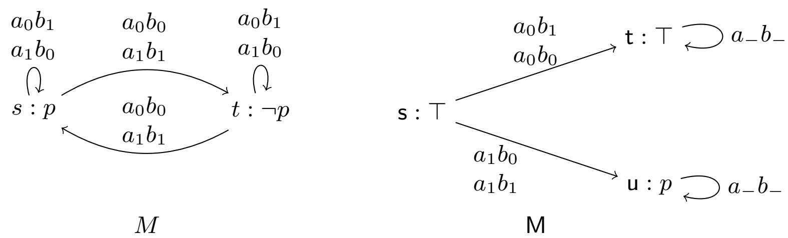

Fig. 1. Model M (left) and action model M (right).

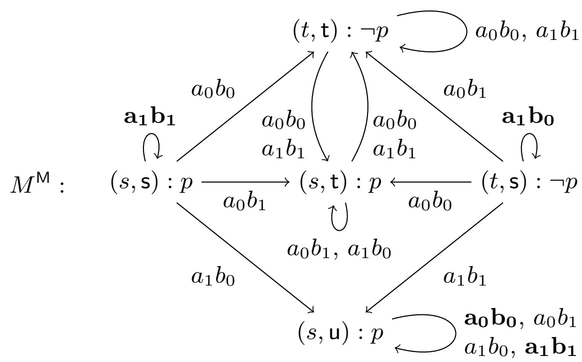

Fig. 2. Updated model M^{M} with added action profiles in bold font.

Example 2. As a continuation of our prize example, consider action model M in Figure 1. In the figure, $a\_b\_$ is a shorthand that the corresponding transition is labelled by all of a_0b_0, a_0b_1, a_1b_0, and a_1b_1.

Action model M describes a policy that prescribes the following modification of the protocol expressed by CGM M. In all states, s or t, if the first agent chooses action a_0, then follow the prize protocol without any modifications (expressed by state t). If the first agent chooses action a_1, then agents enter a state, where each their joint action gets a prize (expressed by state u).

The result of updating CGM M with action model M is updated model M^{M} (Figure 2) that is based on a product of states of M with those states of M that satisfy preconditions. For example, precondition of state s is satisfied by both s and t, and thus we have both (s, s) and (t, s) in the updated model. There is an arrow labelled with an action profile α between some (s, s) and (t, t) if there are arrows labelled with α from s to t and from s to t. Finally, p is satisfied by (s, s) if $p \in L(s)$.

Observe that although in the example for both M and M transitions from each state were defined for each action profile, it is not the case for the corresponding function of M^{M}. The reason for this is that the intersection of transitions from M and M is not guaranteed to include all executable action profiles.

Indeed, in the example, in M_s action profile a_1b_1 takes the agents to a $\neg p$-state, while the same profile in $\mathsf{M_s}$ takes the agents to a p-state. Similarly for action profiles a_1b_0 in states t and s, and a_0b_0 and a_1b_1 in states s and u.

This can be interpreted as the uncertainty (or a conflict) agents may have when a new modification contradicts the existing protocol. We deal with such situations by making the agents remain in the current state in the cases of such uncertainty. In other words, we follow the rule that says *when in doubt, remain where you are*. On the level of updated models this means that for all action profiles α that are not defined, we put $out^{M^{\mathsf{M}}}((s, \mathsf{s}), \alpha) = (s, \mathsf{s})$. That is why in M^{M} action profiles a_1b_0 and a_1b_1 (in bold font) loop back to states (t, s) and (s, s) correspondingly. Moreover, action profiles a_0b_0 and a_1b_1 loop back to (s, u). In this paper, we will write labels of added self-loops in bold font.

Of course, our approach to managing these conflicts is quite conservative, and one can imagine more radical ways of updating a CGM. We leave the exploration of such alternatives for future work.

All in all, action model M updates agents' strategic abilities by taking into account what they actually can achieve in a given CGM M. Thus, in Figure 2 we can indeed see that in state (s, s) action a_0 by the first agents leads to agents executing the same protocol as described by CGM M (states (s, t) and (t, t) in the updated model). On the other hand, contrary to the situation in the initial model, now in the updated model agents can reach state (s, u), where they always receive prizes. Formally, we can write $M_s \not\models \langle\!\langle \{a, b\} \rangle\!\rangle [\![\{a, b\}]\!] p$ and $M_s \models [\mathsf{M_s}] \langle\!\langle \{a, b\} \rangle\!\rangle [\![\{a, b\}]\!] p$, where construct $[\mathsf{M_s}]$ means execution of action model M with the actual state s.

3.2 Syntax and Semantics of Coalition Action Modal Logic

Definition 5. *A* coalitional action model M*, or an* action model*, is a tuple* $(\mathsf{S}, \mathsf{Act}, \mathsf{act}, \mathsf{out}, \mathsf{pre})$*, where* S *is a finite non-empty set of states,* Act *is a non-empty set of actions, act* $: A \times \mathsf{S} \to 2^{\mathsf{Act}} \setminus \emptyset$ *assigns to each agent and each state a non-empty set of actions. Definitions related to action profiles are the same as in the definition of CGM. Function* out *is a* partial function *that maps all action profiles executable in a state to a unique output state. Finally,* $\mathsf{pre} : \mathsf{S} \to \mathcal{CL}$ *assigns to each state a formula of coalition logic. We denote action model* M *with a designated state* s *by* $\mathsf{M_s}$*.*

Definition 6. *The* language of coalition action model logic $\mathcal{CAML}$ *is given recursively by the following grammar:*

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \langle\!\langle C \rangle\!\rangle \varphi \mid [\pi]\varphi$$
$$\pi ::= \mathsf{M_s} \mid (\pi \cup \pi)$$

where $[\pi]\varphi$ *is read 'after execution of* π*,* φ *is true', and the union operator stands for a non-deterministic choice. Dual* $\langle \pi \rangle \varphi$ *is defined as* $\neg[\pi]\neg\varphi$*.*

Definition 7. *Let* $M_s = (S, Act, \ act, \ out, L)$ *be a pointed CGM and* $\mathsf{M_s} = (\mathsf{S}, \mathsf{Act}, \mathsf{act}, \mathsf{out}, \mathsf{pre})$ *be a coalitional action model. The* semantics of CAML *extends the semantics of CL in Definition 2 with the following:*

$$
\begin{array}{ll}
M_s \models [\mathsf{M}_\mathsf{s}]\varphi & \textit{iff } M_s \models \mathsf{pre}(\mathsf{s}) \textit{ implies } M^{\mathsf{M}}_{(s,\mathsf{s})} \models \varphi \\
M_s \models \langle \mathsf{M}_\mathsf{s}\rangle\varphi & \textit{iff } M_s \models \mathsf{pre}(\mathsf{s}) \textit{ and } M^{\mathsf{M}}_{(s,\mathsf{s})} \models \varphi \\
M_s \models [\pi \cup \rho]\varphi & \textit{iff } M_s \models [\pi]\varphi \textit{ and } M_s \models [\rho]\varphi \\
M_s \models \langle \pi \cup \rho\rangle\varphi & \textit{iff } M_s \models \langle\pi\rangle\varphi \textit{ or } M_s \models \langle\rho\rangle\varphi
\end{array}
$$

The updated model M^{M} *is a tuple* $(A, S^{M^{\mathsf{M}}}, Act, act, out^{M^{\mathsf{M}}}, L)$*, where*

$$S^{M^{\mathsf{M}}} = \{(s,\mathsf{s}) \mid s \in S, \mathsf{s} \in \mathsf{S}, \textit{ and } M_s \models \mathsf{pre}(\mathsf{s})\},$$

$$out^{M^{\mathsf{M}}}((s,\mathsf{s}),\alpha) = \begin{cases} (t,\mathsf{t}) & (t,\mathsf{t}) \in S^{M^{\mathsf{M}}}, out(s,\alpha) = t \textit{ and } \mathsf{out}(\mathsf{s},\alpha) = \mathsf{t} \\ (s,\mathsf{s}), & \textit{otherwise.} \end{cases}$$

According to the definition of an updated model, we assume that action models do not grant agents new actions, and, moreover, the valuation of propositional variables remains the same. Thus, action models can be viewed as policy updates that deal only with agents' strategic abilities, and take into account what agents can actually do in the current CGM. Another point worth mentioning is that in our definition of action models we do not require the function out to be total.

There are many similarities between AML and CAML. In particular, all of the following properties are valid for both logics. Their validity in the case of CAML can be shown by application of the definition of the semantics.

Proposition 1. *All of the following are valid.*

1. $\langle \mathsf{M}_\mathsf{s}\rangle\varphi \to [\mathsf{M}_\mathsf{s}]\varphi$
2. $[\mathsf{M}_\mathsf{s}]p \leftrightarrow (\mathsf{pre}(\mathsf{s}) \to p)$
3. $[\mathsf{M}_\mathsf{s}]\neg\varphi \leftrightarrow (\mathsf{pre}(\mathsf{s}) \to \neg[\mathsf{M}_\mathsf{s}]\varphi)$
4. $[\mathsf{M}_\mathsf{s}](\varphi \wedge \psi) \leftrightarrow ([\mathsf{M}_\mathsf{s}]\varphi \wedge [\mathsf{M}_\mathsf{s}]\psi)$
5. $[\pi \cup \rho]\varphi \leftrightarrow [\pi]\varphi \wedge [\rho]\varphi$

The first item states that there is only one way to execute an action model. This is similar to public announcements [19]. Note, however, that in general, $\langle\pi\rangle\varphi \to [\pi]\varphi$ is not valid, since $\langle\pi\rangle\varphi$ is executed non-deterministically. The second property shows that updating a model does not affect propositional variables. Interaction between action models and negation is captured by the third item. Property number four states distributivity of action model updates over conjunction. Finally, the fifth item shows how we can get rid of the union.

Even though Proposition 1 shows that AML and CAML have much in common, the logics are different in a very crucial way. Items two, three, and four of the proposition, interpreted as AML formulas, in conjunction with an interaction principle for AML action models and knowledge modality, constitute *reduction axioms* of AML. This means, that in the context of AML, each formula with an action model can be equivalently rewritten into a formula without it, thus showing that AML is as expressive as epistemic logic, and providing a completeness proof for AML 'for free' (see more on this [11, Sections 6,7, and 8]). We show in the next section that this is not true for CAML.

4 Expressivity

In this section, we argue that, unlike the case of DEL, CAML is strictly more expressive than CL, and thus no reduction axioms are possible. Apart from that, we briefly compare CAML to alternating-time temporal logic (ATL) [5].

Definition 8. *Let φ and ψ be formulas of a language interpreted on CGMs. We say that they are* equivalent *if for all pointed CGMs M_s it holds that $M_s \models \varphi$ iff $M_s \models \psi$.*

Definition 9. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two languages. We say that $\mathcal{L}_1$* is at least as expressive as *$\mathcal{L}_2$ ($\mathcal{L}_2 \leqslant \mathcal{L}_1$) if and only if for all $\varphi \in \mathcal{L}_2$ there is an equivalent $\psi \in \mathcal{L}_1$. If $\mathcal{L}_1$ is* not *at least as expressive as $\mathcal{L}_2$, we write $\mathcal{L}_2 \nleqslant \mathcal{L}_1$. If $\mathcal{L}_2 \leqslant \mathcal{L}_1$ and $\mathcal{L}_1 \nleqslant \mathcal{L}_2$, we write $\mathcal{L}_2 < \mathcal{L}_1$ and say that $\mathcal{L}_1$* is strictly more expressive than *$\mathcal{L}_2$. Finally, if $\mathcal{L}_1 \nleqslant \mathcal{L}_2$ and $\mathcal{L}_2 \nleqslant \mathcal{L}_1$, we say that $\mathcal{L}_1$ and $\mathcal{L}_2$ are incomparable.*

Theorem 2. *$\mathcal{CL} < \mathcal{CAML}$.*

Proof. That $\mathcal{CL} \leqslant \mathcal{CAML}$ follows from the fact that $\mathcal{CL} \subset \mathcal{CAML}$. To show that $\mathcal{CAML} \nleqslant \mathcal{CL}$, consider models M_s from and N_s from Figure 3. In the models, $a_{-}b_{-}$ is a shorthand for all of a_0b_0, a_0b_1, a_1b_0, and a_1b_1. Observe that the two models are quite similar and the difference is that in M_s the agents can force $\neg p$ if they choose actions labelled with the same number, e.g. a_0b_0, and in N_s the agents can force $\neg p$ is they choose actions labelled with different numbers, e.g. a_0b_1. It is easy to check that the two models are bisimilar, and thus satisfy the same formulas of $\mathcal{CL}$ by Theorem 1.

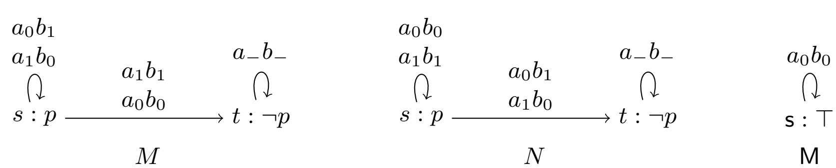

Fig. 3. Models M (left), N (middle), and action model M (right).

Now consider action model M in Figure 3. The action model has only one state with the precondition $\top$ and one self-loop labelled with a_0b_0. The results of updating M_s and N_s with M_s are presented in Figure 4.

It is clear that $M_s \models \langle \mathsf{M}_\mathsf{s} \rangle \langle\!\langle \{a, b\} \rangle\!\rangle \neg p$ and $N_s \not\models \langle \mathsf{M}_\mathsf{s} \rangle \langle\!\langle \{a, b\} \rangle\!\rangle \neg p$. Thus we have that, first, no formula of $\mathcal{CL}$ can distinguish M_s and N_s, and, second, that formula $\langle \mathsf{M}_\mathsf{s} \rangle \langle\!\langle \{a, b\} \rangle\!\rangle \neg p$ of $\mathcal{CAML}$ distinguishes the models. Hence, $\mathcal{CL} \leqslant \mathcal{CAML}$. □

From Theorem 2 it follows that there cannot be any reduction axioms for CAML. There is, however, yet another interesting corollary. In the proof, we

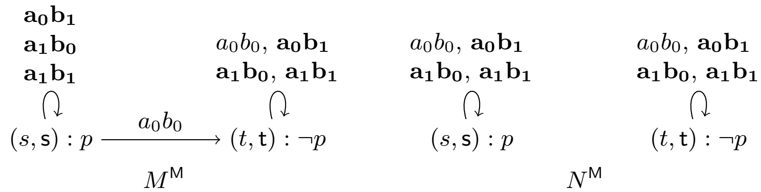

Fig. 4. Updated models M^{M} (left) and N^{M} (right) with added action profiles in bold font.

started with two bisimilar models, and the results of updating them with the same coalitional action model turned out to be not bisimilar. This is quite different from DEL, where updates with action models preserve bisimulation [11, Proposition 6.21].

Corollary 1. *Coalitional action models do not preserve bisimulation.*

Now we turn to the comparison of CAML and ATL [5], with the latter being, probably, the most well-known logic for reasoning about strategic abilities. Other notable examples of such logics include ATL* [5] and strategy logic (SL) [16]. ATL extends CL with temporal operators $\mathsf{X}\varphi$ for 'φ is true in the next step', $\mathsf{G}\varphi$ for 'φ is alway true', and $\psi\mathsf{U}\varphi$ for 'ψ is true until φ'. Due to the lack of space we do not present ATL here, and the the interested reader can find more details in the cited literature [5, 1, 2].

Theorem 3. *$\mathcal{CAML}$ and $\mathcal{ATL}$ are incomparable.*

Proof. We omit the proof for brevity and just give some general intuitions of the main points. First, to argue that $\mathcal{CAML} \not\leq \mathcal{ATL}$, we can reuse the proof of Theorem 2 without any modification with only mentioning that for bisimilar models, Theorem 1 can be extended to $\mathcal{ATL}$ [1]. Second, to show that $\mathcal{ATL} \not\leq \mathcal{CAML}$ we can consider an $\mathcal{ATL}$ formula with Until modality. Then we assume towards a contradiction that there is a $\mathcal{CAML}$ formula of some finite size n that is equivalent to the $\mathcal{ATL}$ formula with Until. After that, we can construct two models of size greater than n so the the $\mathcal{ATL}$ formula can spot a difference with the help of Until, and the $\mathcal{CAML}$ formula does not have enough 'depth' to spot the difference. $\square$

5 Model Checking

Now we show that the model checking problem for CAML is *PSPACE*-complete. It is known that model checking coalition logic can be done in polynomial time, and, thus, in the case of CAML, we have to pay for increased expressivity with higher complexity. This is similar to the situation with DEL, where model checking epistemic logic can be done in polynomial time [14], and the complexity of model checking action model logic is *PSPACE*-complete [8].

Theorem 4. *The model checking problem for CAML is PSPACE-complete.*

Proof. To show that the model checking problem for CAML is in *PSPACE*, we present Algorithm 1. Boolean cases and the case of coalition modalities take polynomial time and we omit them for brevity. For an overview of model checking of strategic logics, including CL, see [10].

Algorithm 1 An algorithm for model checking CAML

procedure $\mathrm{MC}(M, s, \varphi)$
  case $\varphi = [\mathsf{M_s}]\psi$
    if $\mathrm{MC}(M, s, \mathsf{pre}(\mathsf{s}))$ **then**
      return $\mathrm{MC}(M^{\mathsf{M}}, (s, \mathsf{s}), \psi)$
    else
      return *true*
  case $\varphi = [\pi \cup \rho]\psi$
    return $\mathrm{MC}(M, s, [\pi]\psi)$ and $\mathrm{MC}(M, s, [\rho]\psi)$

The algorithm follows the semantics and its correctness can be shown via induction on φ. Now we argue that the algorithm takes at most polynomial space. The interesting case here is $\varphi = [\mathsf{M_s}]\psi$. Since preconditions are formulas of coalition logic, $\mathrm{MC}(M, s, \mathsf{pre}(\mathsf{s}))$ is computed in polynomial time, and hence space. The size of updated model M^{M} is bounded by $\mathcal{O}(|M| \times |\mathsf{M}|) \leqslant \mathcal{O}(|M| \times |\varphi|)$. Finally, since there at most $|\varphi|$ symbols in φ, the total space required by $\mathrm{MC}(M, s, \varphi)$ is bounded by $\mathcal{O}(|M| \times |\varphi|^2)$.

To show hardness, we take the *PSPACE*-hardness proof of the model checking problem for AML [8] as a starting point, and adapt the technique to CGMs and coalitional action models. The main difficulty we face here is that we need to fine-tune models and action models used in the proof in order to ensure that *out* functions behave as expected.

We use the classic reduction from the satisfiability of quantified Boolean formula (QBF) that is known to be *PSPACE*-complete. Also, without loss of generality, we assume that our QBFs have $2k$ variables with alternating quantifiers. See more on satisfiability of such QBFs in [7, p. 83].

For a given QBF $\Psi := \forall x_1 \exists x_2 \dots \forall x_{2k-1} \exists x_{2k} \psi(x_1, \dots, x_{2k})$ we construct in polynomial time a CGM M_s over $A = \{a\}$, action models $\mathsf{AddChain}i_{\mathsf{s}_0^{\mathsf{i}}}$ for all x_i, action model $\mathsf{Copy_t}$, and a formula of $\mathcal{CAML}$ ψ' such that

$$\Psi \text{ is satisfiable iff}$$
$$M_s \models [\mathsf{AddChain1}_{\mathsf{s}_0^1} \cup \mathsf{Copy_t}]\langle \mathsf{AddChain2}_{\mathsf{s}_0^2} \cup \mathsf{Copy_t}\rangle \dots$$
$$[\mathsf{AddChain}(2k-1)_{\mathsf{s}_0^{2k-1}} \cup \mathsf{Copy_t}]\langle \mathsf{AddChain}2k_{\mathsf{s}_0^{2k}} \cup \mathsf{Copy_t}\rangle \psi'.$$

Model M is a tuple (S, Act, act, out, L), where $S = \{s_i \mid 0 \leqslant i \leqslant 2k+1\}$, $Act = \{a_i \mid 0 \leqslant i \leqslant 2k\}$, $act(a, s_i) = Act$ for $0 \leqslant i \leqslant 2k$, and $act(a, s_{2k+1}) = \{a_0\}$, $out(s_i, \alpha) = s_{i+1}$ for $0 \leqslant i \leqslant 2k$, and $out(s_{2k+1}, \alpha) = s_{2k+1}$, and $\{x_i\} = L(s_i)$ for $0 \leqslant i \leqslant 2k+1$. The model is a chain of states of length $2k+1$ such that each next step is reachable via actions a_0, ...,a_{2k} of agent a, and there is a self-loop labelled with a_0 in the last state s_{2k+1}. Each state satisfies exactly one propositional variable x_i.

Coalitional action model $\mathsf{AddChain}i$ is a tuple $(\mathsf{S}, \mathsf{Act}, \mathsf{act}, \mathsf{out}, \mathsf{L})$, where $\mathsf{S} = \{\mathsf{s}^i_j \mid 0 \leqslant j \leqslant i\} \cup \{\mathsf{s}^i_*\}$, $\mathsf{Act} = \{a_i \mid 0 \leqslant i \leqslant 2k\}$, $\mathsf{act}(a, \mathsf{s}^i_j) = \{a_l \mid 1 \leqslant l \leqslant i\}$ for $j \neq i$ and $j \neq 0$, $\mathsf{act}(a, \mathsf{s}^i_0) = \mathsf{Act}$, $\mathsf{act}(a, \mathsf{s}^i_*) = \{a_l \mid 0 \leqslant l \leqslant 2k \text{ and } l \neq i\}$, $\mathsf{out}(\mathsf{s}^i_j, a_l) = \mathsf{s}^i_{j+1}$ for $0 \leqslant j < i$, $1 \leqslant l \leqslant i$ and $l \neq 0$, $\mathsf{out}(\mathsf{s}^i_0, a_l) = \mathsf{s}^i_*$ for $l = 0$ and $l > i$, $\mathsf{out}(\mathsf{s}^i_i, a_i) = \mathsf{s}^i_i$, $\mathsf{out}(\mathsf{s}^i_*, a_j) = \mathsf{s}^i_*$ for $0 \leqslant j \leqslant 2k$ and $j \neq i$, $\mathsf{pre}(\mathsf{s}^i_j) = x_j$ for $0 \leqslant j \leqslant i$, and $\mathsf{pre}(\mathsf{s}^i_*) = \neg x_0$. Action model $\mathsf{AddChain}i$ is a chain of length i where each next state is reachable via all actions a_i excluding a_0, the final state in the chain has a self loop labelled with a_i, and a special state s^i_* is reachable from the first state of the chain via a_0 and all a_j such that $j > i$. The intuition behind the action models is that $\mathsf{AddChain}i$'s add chains of length i to M_s meaning that variable x_i has been set to 1. Moreover, all other chains that were already in a CGM are not affected.

Coalitional action model Copy is a tuple $(\mathsf{S}, \mathsf{Act}, \mathsf{act}, \mathsf{out}, \mathsf{L})$, where $\mathsf{S} = \{\mathsf{t}\}$, $\mathsf{Act} = \{a_i \mid 0 \leqslant i \leqslant 2k\}$, $\mathsf{act}(a, \mathsf{t}) = \mathsf{Act}$, $\mathsf{out}(\mathsf{t}, \alpha) = \mathsf{t}$, and $\mathsf{pre}(\mathsf{t}) = \top$. Action model Copy just copies a given model so that no new chain appears meaning that the current x_i has been set to 0.

Finally, we translate $\psi(x_1, \dots, x_{2k})$ by substituting every x_i with $(\langle\!\langle a \rangle\!\rangle)^i [\![a]\!] x_i$, where $(\langle\!\langle a \rangle\!\rangle)^i$ is a stack of size i of $\langle\!\langle a \rangle\!\rangle$'s. In the resulting translated formula, subformula $(\langle\!\langle a \rangle\!\rangle)^i [\![a]\!] x_i$ holds if in a model there is a chain of length i with a loop at the end. This means that variable x_i has been set to 1.

As an example, consider a QBF $\forall x_1 \exists x_2 (x_1 \to x_2)$. We translate the formula into a CAML formula

$$[\mathsf{AddChain1}_{\mathsf{s}^1_0} \cup \mathsf{Copy}_\mathsf{t}]\langle \mathsf{AddChain2}_{\mathsf{s}^2_0} \cup \mathsf{Copy}_\mathsf{t}\rangle(\langle\!\langle a \rangle\!\rangle [\![a]\!] x_1 \to \langle\!\langle a \rangle\!\rangle\langle\!\langle a \rangle\!\rangle [\![a]\!] x_2).$$

The corresponding model M and action models $\mathsf{AddChain1}$, $\mathsf{AddChain2}$, and Copy are presented in Figure 5.

According to the semantics,

$$M_s \models [\mathsf{AddChain1}_{\mathsf{s}^1_0} \cup \mathsf{Copy}_\mathsf{t}]\langle \mathsf{AddChain2}_{\mathsf{s}^2_0} \cup \mathsf{Copy}_\mathsf{t}\rangle(\langle\!\langle a \rangle\!\rangle [\![a]\!] x_1 \to \langle\!\langle a \rangle\!\rangle\langle\!\langle a \rangle\!\rangle [\![a]\!] x_2)$$

if and only if

$$M_s \models [\mathsf{AddChain1}_{\mathsf{s}^1_0}]\langle \mathsf{AddChain2}_{\mathsf{s}^2_0} \cup \mathsf{Copy}_\mathsf{t}\rangle(\langle\!\langle a \rangle\!\rangle [\![a]\!] x_1 \to \langle\!\langle a \rangle\!\rangle\langle\!\langle a \rangle\!\rangle [\![a]\!] x_2)$$

and

$$M_s \models [\mathsf{Copy}_\mathsf{t}]\langle \mathsf{AddChain2}_{\mathsf{s}^2_0} \cup \mathsf{Copy}_\mathsf{t}\rangle(\langle\!\langle a \rangle\!\rangle [\![a]\!] x_1 \to \langle\!\langle a \rangle\!\rangle\langle\!\langle a \rangle\!\rangle [\![a]\!] x_2).$$

In other words, $\langle \mathsf{AddChain2}_{\mathsf{s}^2_0} \cup \mathsf{Copy}_\mathsf{t}\rangle(\langle\!\langle a \rangle\!\rangle [\![a]\!] x_1 \to \langle\!\langle a \rangle\!\rangle\langle\!\langle a \rangle\!\rangle [\![a]\!] x_2)$ should hold in both $M^{\mathsf{AddChain1}}_{(s_0, \mathsf{s}^1_0)}$ and $M^{\mathsf{Copy}}_{(s_0, \mathsf{t})}$. Updated model $M^{\mathsf{AddChain1}}_{(s_0, \mathsf{s}^1_0)}$ is depicted in Figure 6, and model $M^{\mathsf{Copy}}_{(s_0, \mathsf{t})}$ will just copy M, so we do not provide the figure.

Now, for each of $M^{\mathsf{AddChain1}}_{(s_0, \mathsf{s}^1_0)}$ and $M^{\mathsf{Copy}}_{(s_0, \mathsf{t})}$ there must be a subsequent update with either $\mathsf{AddChain2}$ or Copy such that $\langle\!\langle a \rangle\!\rangle [\![a]\!] x_1 \to \langle\!\langle a \rangle\!\rangle\langle\!\langle a \rangle\!\rangle [\![a]\!] x_2$ will hold in the resulting model.

The result of updating M^{Copy} with $\mathsf{AddChain2}$ is shown in Figure 6. Note that $M^{\mathsf{Copy},\mathsf{AddChain2}}_{(s_0, \mathsf{t}, \mathsf{s}^2_0)} \models \langle\!\langle a \rangle\!\rangle [\![a]\!] x_1 \to \langle\!\langle a \rangle\!\rangle\langle\!\langle a \rangle\!\rangle [\![a]\!] x_2$ as the antecedent is not satisfied.

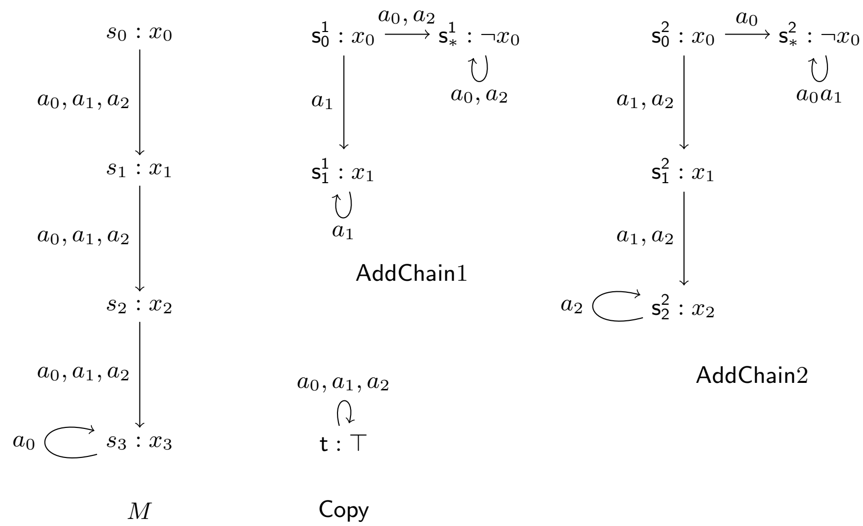

Fig. 5. Model M, and action models $\mathsf{AddChain1}$, $\mathsf{AddChain2}$, and Copy.

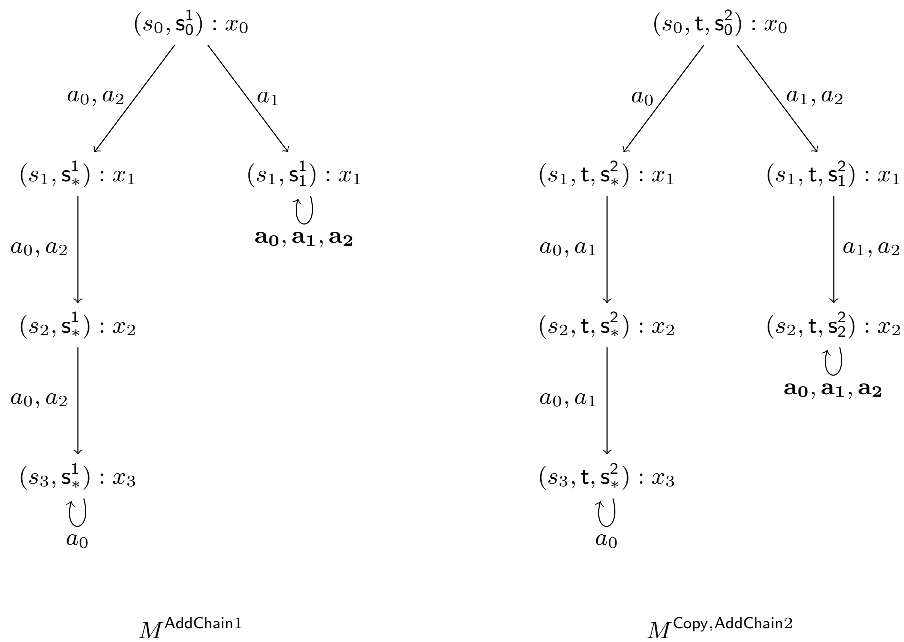

Fig. 6. Updated models $M^{\mathsf{AddChain1}}$ and $M^{\mathsf{Copy},\mathsf{AddChain2}}$ with added action profiles in bold font.

Also observe that $M^{\mathsf{Copy},\mathsf{Copy}}_{(s_0,\mathsf{t},\mathsf{t})}$ would satisfy the formula for the same reason. All in all, this corresponds to setting x_1 to 0 in the original QBF, and thus the QBF will be true irregardless of the value of x_2.

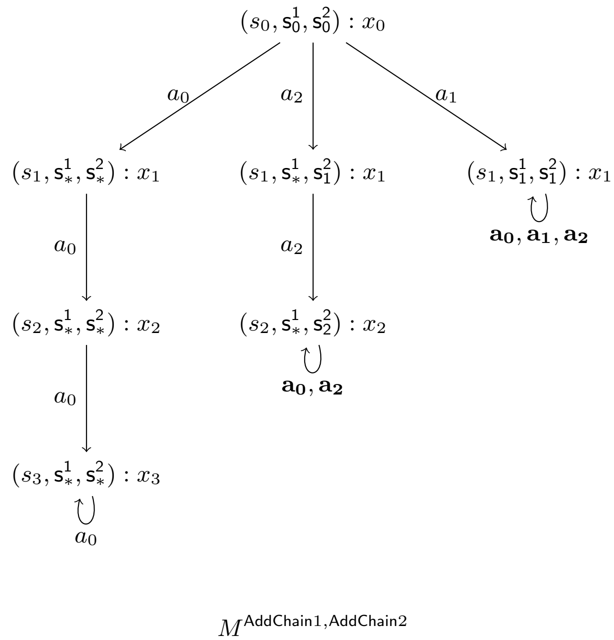

Fig. 7. Updated model $M^{\mathsf{AddChain1},\mathsf{AddChain2}}$ with added action profiles in bold font.

Consider updated model $M^{\mathsf{AddChain1}}_{(s_0,\mathsf{s}_0^1)}$. It has only chains of lengths 1 and 3, and thus we have that $M^{\mathsf{AddChain1}}_{(s_0,\mathsf{s}_0^1)} \models \langle\!\langle a\rangle\!\rangle [\![a]\!] x_1$ and at the same time $M^{\mathsf{AddChain1}}_{(s_0,\mathsf{s}_0^1)} \not\models \langle\!\langle a\rangle\!\rangle\langle\!\langle a\rangle\!\rangle [\![a]\!] x_2$. So, $M^{\mathsf{AddChain1}}_{(s_0,\mathsf{s}_0^1)}$ does not satisfy formula $\langle\!\langle a\rangle\!\rangle [\![a]\!] x_1 \to \langle\!\langle a\rangle\!\rangle\langle\!\langle a\rangle\!\rangle [\![a]\!] x_2$. Hence, updating it with Copy would also result in a model, where the formula is not satisfied. This corresponds to choosing value 1 for x_1 in our QBF, and setting x_2 to 0 will not make the QBF true. However, choosing 1 for x_2 satisfies the formula. In terms of updated models this corresponds to updating $M^{\mathsf{AddChain1}}_{(s_0,\mathsf{s}_0^1)}$ with $\mathsf{AddChain2}$, and the result of such an update is depicted in Figure 7. Note that in Figure 7 we take the connected component that includes state $(s_0, \mathsf{s}_0^1, \mathsf{s}_0^2)$, and we disregard state $(s_1, \mathsf{s}_1^1, \mathsf{s}_*^2)$ that will not be connected to the chosen component. It is clear that $M^{\mathsf{AddChain1},\mathsf{AddChain2}}_{(s_0,\mathsf{s}_0^1,\mathsf{s}_0^2)}$, which corresponds to setting both x_1 and x_2 to 1, satisfies $\langle\!\langle a\rangle\!\rangle [\![a]\!] x_1 \to \langle\!\langle a\rangle\!\rangle\langle\!\langle a\rangle\!\rangle [\![a]\!] x_2$.

Our construction mimics QBFs in the following way. For a universal quantifer $\forall x_i$ we use $[\mathsf{AddChain}i_{\mathsf{s}_0^i} \cup \mathsf{Copy}_\mathsf{t}]$ that corresponds to producing an updated model with a chain of length i, setting x_i to 1, and an updated model without such a chain, setting x_i to 0. In the case of $\exists x_i$, the choice between $\mathsf{AddChain}i_{\mathsf{s}_0^i}$ and Copy_t is existential, which is expressed by $\langle \mathsf{AddChain}i_{\mathsf{s}_0^i} \cup \mathsf{Copy}_\mathsf{t}\rangle$. As a result of such a choice, we will have an updated model with a chain of length i, or an updated model without such a chain. $\square$

Remark 1. Our hardness reduction relied heavily on non-deterministic choice, i.e. constructs $[\pi \cup \rho]$ and $\langle \pi \cup \rho \rangle$. As we have already mentioned in Proposition 1, item five, we can equivalently rewrite formulas with unions to formulas without it. This rewriting, however, can result in a formula of exponential size. We leave the problem of determining hardness of model checking CAML without union open, and conjecture that it is still *PSPACE*-hard. On a similar note, a more complicated construction than the one used in [8] was employed to show that DEL without union is *PSPACE*-hard [13, Theorem 4].

6 Discussion

We presented coalition action model logic (CAML) for reasoning about how agents' abilities change as a result of updating a CGM with a coalitional action model. Even though we took inspiration from DEL, CAML turned out quite different. In particular, CAML is strictly more expressive that the base CL, and thus no reduction axioms possible. We also claimed that CAML is incomparable to ATL, and conjecture that the same holds for other logics for reasoning about strategic abilities, namely ATL* and SL. Finally, we investigated the complexity of the model checking problem for CAML, and showed that it is *PSPACE*-complete.

Since this is the first proposal of DEL-like action models for CGMs, there is a plethora of open questions. First, the non-existence of reduction axioms leaves open the problem of providing a sound and complete axiomatisation of CAML. Moreover, it is also worthwhile to investigate coalitional action models with postconditions, i.e. action models that allow changing valuations of propositional variables. While we expect that postconditions will not affect the complexity of model-checking, expressivity results may turn out to be more surprising. Another avenue of further research is having a more expressive base language than CL. In particular, we plan to use action models with ATL and ATL*. Apart from that, we had to make a design decision that whenever the result of executing an action profile is undefined (or, there is a conflict between the existing model and a proposed modification), then a given system remains in the same state. However, there may be other intuitively natural ways to handle situations like that. Finally, our action models are quite conservative in the sense that they neither grant agents new actions nor revoke any actions. It would be exciting to come up with action models that affect agents' sets of available actions.

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References

1. Ågotnes, T., Goranko, V., Jamroga, W.: Alternating-time temporal logics with irrevocable strategies. In: Samet, D. (ed.) Proceedings of the 11th TARK. pp. 15–24 (2007)
2. Ågotnes, T., Goranko, V., Jamroga, W., Wooldridge, M.: Knowledge and ability. In: van Ditmarsch, H., Halpern, J.Y., van der Hoek, W., Kooi, B. (eds.) Handbook of Epistemic Logic, pp. 543–589. College Publications (2015)
3. Ågotnes, T., van der Hoek, W., Juan A. Rodriguez-Aguilar, C.S., Wooldridge, M.: On the logic of normative systems. In: Veloso, M.M. (ed.) Proceedings of the 20th IJCAI. pp. 1175–1180 (2007)
4. Ågotnes, T., van der Hoek, W., Wooldridge, M.: Robust normative systems and a logic of norm compliance. Logic Journal of the IGPL **18**(1), 4–30 (2010)
5. Alur, R., Henzinger, T.A., Kupferman, O.: Alternating-time temporal logic. Journal of the ACM **49**, 672–713 (2002)
6. Areces, C., Fervari, R., Hoffmann, G.: Relation-changing modal operators. Logic Journal of the IGPL **23**(4), 601–627 (2015)
7. Arora, S., Barak, B.: Computational Complexity: A Modern Approach. CUP (2009)
8. Aucher, G., Schwarzentruber, F.: On the complexity of dynamic epistemic logic. In: Schipper, B.C. (ed.) Proceedings of the 14th TARK (2013)
9. Baltag, A., Moss, L.S.: Logics for epistemic programs. Synthese **139**(2), 165–224 (2004)
10. Bulling, N., Dix, J., Jamroga, W.: Model checking logics of strategic ability: Complexity. In: Dastani, M., Hindriks, K.V., Meyer, J.J.C. (eds.) Specification and Verification of Multi-agent Systems, pp. 125–159. Springer (2010)
11. van Ditmarsch, H., van der Hoek, W., Kooi, B.: Dynamic Epistemic Logic, Synthese Library, vol. 337. Springer (2008)
12. Galimullin, R., Ågotnes, T.: Dynamic coalition logic: Granting and revoking dictatorial powers. In: Ghosh, S., Icard, T. (eds.) Proceedings of the 8th LORI. LNCS, vol. 13039, pp. 88–101. Springer (2021)
13. de Haan, R., van de Pol, I.: On the computational complexity of model checking for dynamic epistemic logic with S5 models. FLAP **8**(3), 621–658 (2021)
14. Halpern, J.Y., Moses, Y.: A guide to completeness and complexity for modal logics of knowledge and belief. Artificial Intelligence **54**(2), 319–379 (1992)
15. Kooi, B., Renne, B.: Arrow update logic. Review of Symbolic Logic **4**(4), 536–559 (2011)
16. Mogavero, F., Murano, A., Vardi, M.Y.: Reasoning about strategies. In: Lodaya, K., Mahajan, M. (eds.) Proceedings of the 30th FSTTCS. LIPIcs, vol. 8, pp. 133–144. Schloss Dagstuhl - Leibniz-Zentrum für Informatik (2010)
17. Pauly, M.: Logic for Social Software. Ph.D. thesis, ILLC, University of Amsterdam, The Netherlands (2001)
18. Pauly, M.: A modal logic for coalitional power in games. Journal of Logic and Computation **12**(1), 149–166 (2002)
19. Plaza, J.: Logics of public communications. Synthese **158**(2), 165–179 (2007)
20. Shoham, Y., Tennenholtz, M.: On the synthesis of useful social laws for artificial agent societies. In: Proceedings of the 10th AAAI (1992)
21. Shoham, Y., Tennenholtz, M.: On social laws for artificial agent societies: Offline design. In: Agre, P.E., Rosenschein, S.J. (eds.) Computational Theories of Interaction and Agency, pp. 597–618. MIT Press (1996)

Quantum Logic for Observation of Physical Quantities

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Abstract. Dynamic quantum logic (DQL) is studied to represent the proposition of dynamism in quantum information theory as traditional quantum logic cannot deal with it. Although DQL includes many important notions for quantum physics and quantum information theory, there are still missing elements. Some concepts of measuring a specific physical quantity cannot be represented in the exiting DQL. In this study, we add a new concept of measurement to DQL, and discuss the property of this new logic.

1 Introduction

Quantum logic (QL) has been studied to handle unique propositions that appear in quantum physics. Moreover, numerous types of logics and structures have been proposed to represent and analyze such propositions [11] [12] [16] [23]. In particular, logic based on *orthomodular lattices*, namely, *orthomodular logic* (OML), has been studied since 1936, proposed by Birkhoff and Von Neumann [10]. An orthomodular lattice is related to the closed subspaces of a Hilbert space, which is a state space of a particle in quantum physics. Instead of these lattices, the *Kripke model* (possible world model) of OML can be used, which is called the *orthomodular-model* (OM-model) [11] [20] [21]. Intuitively, each possible world of an OM-model expresses an one-dimensional subspace of a Hilbert space, corresponding to a *quantum state*. In quantum mechanics, due to the *uncertainty principle*, exact values cannot be simultaneously obtained for a specific set of physical quantities (for example, momentum and position along an axis). This statistical property is the nature of the states of the object and exists independently of an experimenter's knowledge. OML handles the most basic part of this special nature of states. For more details on QL and quantum physics, see [11] [12].

In OML, *static* propositions such as "In this state, A is true" are treated. *Dynamic* propositions such as "After unitary evolution U, A is true at a state" cannot be treated in OML. To deal with dynamic propositions and modality in quantum physics, [2]-[6] have introduced *dynamic quantum logic* (DQL). In DQL, some dynamic concepts are defined as modal symbols. In particular, two dynamism, *unitary transformations (unitary evolutions)* and *projections* onto closed subspaces, have been mainly analyzed. These two dynamisms play a central role in the dynamic notion of quantum mechanics. Unitary transformations are the most basic state changes in quantum mechanics. Intuitively, this change corresponds to the change due to the equation of motion in classical mechanics. On a Hilbert space, these changes are expressed by unitary operators. Changes due to projection are unique to quantum mechanics, and appear when a physical quantity of the state is observed. This concept will be explained later. As the quantum theory progresses, various types of dynamism appear. They are revealed by the analysis of various kinds of operators in a Hilbert space.

Although some modalities in quantum physics have been studied in mathematical logic, logical analysis of dynamic concepts in quantum mechanics is still underway. In particular, there is still room for development in analysis in the direction using *abstract* models. In general, QL has been developed using two primary methods. The first method is research using models that can express almost all properties of Hilbert spaces [12] [14]. In this context, the Hilbert space is often used as a model. The second method is research using a simple model that includes only specific parts of a Hilbert space [10] [24]. Studies using orthomodular lattices formed by observational propositions of a Hilbert space belong to this category. Each of these two methods has its advantages and disadvantages. The former method is suitable for detailed and diverse analysis of quantum mechanics because it can express almost all propositions for the states or values of physical quantities in quantum mechanics. However, it has the disadvantage that logical analysis is difficult because logical symbols and models become quite complex. Although detailed analysis is impossible in the latter method, specific properties can be treated in detail. Further, because simple logical symbols and models are used, it is easy to perform logical analysis and comparison with other logics.

The former method is extremely common when considering propositions about complex notions in quantum mechanics. The second method with complex notions of quantum mechanics has not been studied enough because it cannot handle complex concepts without constructing the model and logic well. To develop an abstract method, it is desirable to develop it without complicating the concepts used as much as possible. For example, when using a binary relation model, it is desirable to only increase the types of relations or formulas, and it is not so preferable to add other complicated mathematical structures. In this study, we add important concepts in quantum mechanics to abstract quantum logic while keeping this constraint.

In quantum physics, the concept of observing the properties of a particle has been extensively studied. Various types of observations are defined and analyzed depending on the observation method and accuracy. Mathematically, positive operator valued measurement (POVM) is widely studied as a general measurement [19]. POVM can be divided into several types depending on what kind of operator is used. Among them, *projection valued measurement* (PVM) is recognized as a basic measurement. In this measurement method, the propositions of a physical quantity are associated with each closed subspace in a Hilbert space or projection operator corresponding to it. The result of the measurement is determined by the probability associated with each projection. After the measurement, the state is projected onto the closed subspace corresponding to the obtained proposition.

In general, physical quantity is represented by an *orthonormal basis* of a Hilbert space. Each physical quantity corresponds to each orthonormal basis, and each value corresponds to each base. Therefore, PVM of strict value (not in range) of a physical quantity is represented by projection onto an orthonormal basis of a Hilbert space. For example, after the value of a physical quantity is observed to be 3, the state is immediately projected onto the base which 3 is assigned.

In mathematical logic, these concepts of measurement have been studied abstractly in some contexts. For example, in [3] and [4], the nature of the observer's knowledge when observing physical quantities is analyzed. These contexts are based on DQL and involve DQL's nature to handle the concept of projection concisely.

Although the notion of projection can be used in DQL, the proposition of projective measurement can only be dealt with in a limited way. Intuitively, the concept of projection used in DQL can only handle the so-called YES-NO judgment of whether or not proposition A holds. This measurement correspond to PVM composed of projections onto the closed subspace that A is true and the orthogonal closed subspace of it. The formula of DQL $[A?]B$ can be translated as "After testing whether or not proposition A holds, if A is true, then B is also true." In this setting, propositions involving the *measurement of specific physical quantity*, such as "After observing one value of a physical quantity M, whatever the value, A is true" cannot be handled because in the present studies of DQL, there is no modal symbol for projection onto specific orthonormal basis. Moreover, the current DQL model does not include the concept of an orthonormal basis. Therefore, DQL cannot express some types of PVM. To overcome this problem, in this study, we add the notion of an orthonormal basis to the model and add new symbols for measurement to the language, and analyze its nature. The formula correspondences with the new modal symbols for important conditions are also given. Similar to the original DQL, these formulas express important elements of a Hilbert space.

We add a new class of propositional variables to the language, and bases are expressed by these variables. Moreover, a new modal symbol is defined as projections onto these bases. The properties of orthogonal bases are represented by axioms and rules that include this symbol. As one of the features of this method, we do not use symbols that specifically specify the state, such as nominal, since we want to keep the concept as simple as possible.

As a model, we adopt the model of DQL and construct a new model by adding abstract concepts of orthonormal basis to it. Intuitively, a model for DQL is constructed by adding some conditions of Hilbert spaces to a basic model for *dynamic logic* [13] [15]. This model does not introduce all the properties of Hilbert space but expresses only some of the properties that are suitable for binary relations. [2]. Some cumbersome conditions for Hilbert spaces are not added to the model, as these conditions do not appear in the main argument.

Although most conditions are the same as in [2] but modified slightly to be suitable for handling in this study. This change does not affect the outline of the discussion.

In section 2, we review the definition and concept of DQL. In section 3, we discuss the main topics. We discuss in detail why normal DQL is incompatible with some concepts of measurement, and discuss how to solve this problem by adding new modal symbols for measurement and constructing a new model. In section 4, some remarks and future works are discussed.

2 Basics

In this section, we review the definition and concept of DQL using mainly the basic notations in [2]. We omit some details here; see [2]-[6] for full details about DQL. We use almost the same language as dynamic logic but without Kleene star. We add the symbol † for the Hermitian conjugate of a transformation. We also add the modal symbol $\Box$ to denote non-orthogonality. We use $p, q, \ldots$ to denote propositional variables and $A, B, C, \ldots$ to denote composite formulas. We use $U, V, \ldots$ to denote *variables for unitary evolution* and π to denote composite *actions*. We use $\mathcal{U}$ to denote the set of all variables for unitary evolution. Formulas and actions are defined below.

$$A ::= p \mid \bot \mid \neg A \mid A \wedge B \mid \Box A \mid [\pi]A$$
$$\pi ::= U \mid \pi^{\dagger} \mid \pi \cup \pi \mid \pi ; \pi \mid A?$$

We use the following abbreviations: $A \vee B = \neg(\neg A \wedge \neg B)$, $A \to B = \neg A \vee B$、$A \leftrightarrow B = (A \to B) \wedge (B \to A)$, $\Diamond A = \neg \Box \neg A$、$\langle \pi \rangle A = \neg[\pi]\neg A$. We write $A[p/B]$ to mean that all appearances of p in A are replaced by B.

A *dynamic frame* is defined as a triple $\langle X, \{\xrightarrow{Y?}\}_{Y \subseteq X}, \{\xrightarrow{U}\}_{U \in \mathcal{U}} \rangle$. X is a nonempty set. $\{\xrightarrow{Y?}\}_{Y \subseteq X}$ is a set of binary relations $Y?(Y \subseteq X)$ on X. $\{\xrightarrow{U}\}_{U \in \mathcal{U}}$ is a set of binary relations $U \in \mathcal{U}$ on X. That is, for every $Y \subseteq X$ and $U \in \mathcal{U}$, we define a binary relation on X. Moreover, we use the same symbols of unitary evolutions in formulas and binary relations. For any relation R and for every $x, y \in X$, we write $x(R)y$ if $(x, y) \in R$. We define the composite relation $R; R'$ by $R; R' = \{x, y \in X \mid$ there exists $z \in X$ such that $x(R)z$ and $z(R')y\}$. We also We introduce a binary relation $\not\perp$ on X as follows:

$x \not\perp y \;\Leftrightarrow$ there exists $Y \subseteq X$ such that $x(Y?)y$ or $y(Y?)x$.

We write $x \perp y$ if not $x \not\perp y$. We write $x \perp Y$ if for all $y \in Y$, $x \perp y$ where $x \in X$ and $Y \subseteq X$. Given $Y \subseteq X$, we define the set $Y^{\perp} = \{x \in X \mid x \perp Y\}$. We say that Y is *testable* if $Y^{\perp\perp} = Y$. We write $Y \perp Z$ if for all $y \in Y$ and $z \in Z$, $y \perp z$.

We define a *dynamic quantum frame* $\langle X, \{\xrightarrow{Y?}\}_{Y \subseteq X}, \{\xrightarrow{U}\}_{U \in \mathcal{U}} \rangle$ by adding the following conditions to a dynamic frame. In these conditions, new binary relations in the form of $R^{\dagger}$ appear. However, it can be seen that these new relations are uniquely determined by $\{\xrightarrow{Y?}\}_{Y \subseteq X}$ and $\{\xrightarrow{U}\}_{U \in \mathcal{U}}$ from the conditions. This is the same as the conjugate $P^\dagger$ of the operator P being uniquely determined in a Hilbert space. Therefore, we do not express these relations in the expression of a dynamic quantum frame $\langle X, \{\xrightarrow{Y?}\}_{Y\subseteq X}, \{\xrightarrow{U}\}_{U\in\mathcal{U}}\rangle$.

1. There is no $x, y \in X$ such that $x(\emptyset?)y$. For all $x \in X$, $x(X?)x$.
2. For all $x, y, z \in X$, if $x(Y?)y$ and $x(Y?)z$, then $y = z$. (Partial functionality of $P?$)
3. If $x \in Y$, then $x(Y?)x$. (Adequacy)
4. For all $x, y \in X$, if $Y \subseteq X$ is testable and $x(Y?)y$, then $y \in Y$. (Repeatability)
5. For all $Y, Z \subseteq X$, if Y and Z are testable and $Y?; Z? = Z?; Y?$, then $Y?; Z? = (Y \cap Z)?$. (Compatibility)
6. Let (R) be $(Y?), (U)$, or $(U^\dagger)$. If $x(R)y$ and $y \not\perp z$, then there exists $w \in X$ such that $z(R^\dagger)w$ and $w \not\perp x$. (Adjointness)
7. For all $x \in X$ and U, $\exists! y \in X$ such that $x(U)y$. (Functionality for U)
8. For all $x \in X$ and $U^\dagger$, $\exists! y \in X$ such that $x(U^\dagger)y$. (Functionality for $U^\dagger$)
9. For all $x, y \in X$, $x(U)y$ iff $y(U^\dagger)x$. (Bijectivity)
10. For all $x, y \in X$ and $Y \subseteq X$, $x(Y?)y$ iff $x(Y?^\dagger)y$.
11. For all $x, y \in X$, there exists $z \in X$ such that $x \not\perp z$ and $z \not\perp y$. (Universal accessibility)

This definition is almost the same as the definition in [2] but modified slightly to be suitable for handling in this study, and important properties derived from existing conditions are explicitly included. These conditions represent some nature of a Hilbert space [2].

From the definition of $\not\perp$ and condition 1, $\not\perp$ is confirmed as a symmetric and reflective relation.

We define a *dynamic quantum model* (DQM) $\langle X, \{\xrightarrow{Y?}\}_{Y\subseteq X}, \{\xrightarrow{U}\}_{U\in\mathcal{U}}, V\rangle$, where $\langle X, \{\xrightarrow{Y?}\}_{Y\subseteq X}, \{\xrightarrow{U}\}_{U\in\mathcal{U}}\rangle$ is a dynamic quantum frame and V is a function mapping each propositional variable p to a subset of X. We define the sets $\|A\|$ on a DQM by induction on the composition of A as follows. We write $x(A?)y$ if $\|A\| = Y \subseteq X$ and $x(Y?)y$.

$\|p\| = V(p)$
$\|\bot\| = \emptyset$
$\|A \wedge B\| = \|A\| \cap \|B\|$
$\|\neg A\| = \|A\|^c$
$\|\Box A\| = \{x \in X |$ for all $y \in X$, if $x \not\perp y$, then $y \in \|A\|\}$
$\|[A?]B\| = \{x \in X |$ for all $y \in X$, if $x(A?)y$, then $y \in \|B\|\}$
$\|[U]A\| = \{x \in X |$ for all $y \in X$, if $x(U)y$, then $y \in \|A\|\}$
$\|[\pi_1; \pi_2]A\| = \|[\pi_1][\pi_2]A\|$
$\|[\pi_1 \cup \pi_2]A\| = \|[\pi_1]A\| \cap \|[\pi_2]A\|$
$\|[B?^\dagger]A\| = \|[B?]A\|$
$\|[U^\dagger]A\| = \{x \in X |$ for all $y \in X$, if $x(U^\dagger)y$, then $y \in \|A\|\}$
$\|[(\pi_1; \pi_2)^\dagger]A\| = \|[\pi_2^\dagger; \pi_1^\dagger]A\|$
$\|[(\pi_1 \cup \pi_2)^\dagger]A\| = \|[\pi_1^\dagger \cup \pi_2^\dagger]A\|$

$\|[\pi^{\dagger\dagger}]\|A = \|[\pi]A\|$

We say that formula A is *true* at $x \in X$ if $x \in \|A\|$. We say that A is *false* at $x \in X$ if $x \notin \|A\|$. We say that A is *valid* in a DQM $\langle X, \{\xrightarrow{Y?}\}_{Y \subseteq X}, \{\xrightarrow{U}\}_{U \in \mathcal{U}}, V\rangle$ if for all $x \in X$, A is true at x. We say A is a *testable formula* if $\|A\|$ is testable in all DQMs. We say that a DQM is *T-complete* if for all testable sets $Y \subseteq X$ there exists A such that $\|A\| = Y$. In the next section, we need this concept to be able to represent all testable sets by formulas.

Axioms and rules for dynamic quantum logic is introduced as follows [2]-[6], based on the traditional modal logic [17] [18]. In this study, we call this system **PDQL** (propositional dynamic quantum logic).

All the axioms and rules of classical dynamic logic
(Necessitation Rule): If A is provable, then infer $[\pi]A$
(Kripke Axiom): $[\pi](A \to B) \to ([\pi]A \to [\pi]B)$
(Test Generalization): If $A \to [C?]B$ is provable for all C, then infer $A \to \Box B$
(Testability Axiom): $\Box A \to [B?]A$
(Partial Functionality): $\neg[A?]B \to [A?]\neg B$
(Adequacy): $A \wedge B \to \langle A?\rangle B$
(Repeatability): $[A?]A$ for all testable formulas A
(Universal Accessibility): $\langle\pi\rangle\Box\Box A \to [\pi']A$
(Unitary Functionality): $\neg[U]A \leftrightarrow [U]\neg A$
(Unitary Bijectivity 1): $A \leftrightarrow [U; U^{\dagger}]A$
(Unitary Bijectivity 2): $A \leftrightarrow [U^{\dagger}; U]A$
(Adjointness): $A \to [\pi]\Box\langle\pi^{\dagger}\rangle\Diamond A$
(Substitution Rule): If A is provable, then infer $A[p/B]$
(Compatibility Rule): For all testable formulas A, B and every propositional variable p which does not appears in A, B, if $\langle A?; B?\rangle p \to \langle B?; A?\rangle p$ is provable, then infer $\langle A?; B?\rangle p \to \langle (A \wedge B)?\rangle p$

From the universal accessibility of a model, $\Box\Box A$ means that A is true at all $x \in X$. We require the following lemma for the next section.

Lemma 1. *In any dynamic quantum frame, if $x \not\perp y$, $y \in Y$ and Y is testable, then there exists $z \in Y$ such that $x(Y?)z$.*

Proof. From adequacy, $y(Y?)y$. From the adjointness of $(Y?)$, there exists $z \in X$ such that $x(Y?)z$ and $z \not\perp y$. From the testability of Y and repeatability, $z \in Y$. □

Furthermore, we use the following abbreviations of formulas for convenience:

$\sim A = \Box\neg A$
$T(A) = \Box\Box(\sim\sim A \to A)$

$\sim$ represents *quantum negation.* In any DQM, $\|A\|$ is testable if and only if $\|A\| = \|\sim\sim A\|$. Because $A \to \sim\sim A$ is always true, if $\sim\sim A \to A$ is true at all $x \in X$, then $\|A\|$ is testable. Therefore, $T(A)$ means that A is testable.

3 Modality for Measurement

In this section, we introduce a new modal operator that represents PVM in special cases. We deal with a PVM consisting of all the projections onto the eigenstates associated with a physical quantity M. In this situation, when a physical quantity M is measured, the state is projected onto one element of an orthonormal basis that corresponds to M in a Hilbert space. Although this movement of a state is a fundamental concept in quantum physics, a proposition such as "After a measurement of M, A is true" cannot be represented by the formulas given in section 2. A projection to a closed subspace of a Hilbert space is represented by the modal symbol $[A?]$. Therefore, intuitively, "After a measurement of M, A is true" may be represented by $[B_1? \cup B_2? \cup B_3? \cup \ldots]A$ where B_i corresponds to an orthonormal basis of M in a Hilbert space. Intuitively, this expression has three problems. Firstly, in the DQM, we did not introduce the notion of an orthonormal basis. Secondly, we cannot show that B_i corresponds to an orthonormal basis of M because the propositional variables could be any subset of a Hilbert space. Thirdly, if an orthonormal basis has infinite elements, the set $\{B_1, B_2, B_3, \ldots\}$ will be an infinite set. However, the infinite chain $B_1 \cup B_2 \cup \ldots$ is not allowed.

As a solution to the first problem, we introduce a set for an orthonormal basis in a dynamic quantum frame. $\mathcal{P}(X)$ denotes the power set of X. Given a dynamic quantum frame $\langle X, \{\xrightarrow{Y?}\}_{Y \subseteq X}, \{\xrightarrow{U}\}_{U \in \mathcal{U}}\rangle$, we say that a set $Ob \subseteq \mathcal{P}(X)$ is an *orthonormal basis* of $\langle X, \{\xrightarrow{Y?}\}_{Y \subseteq X}, \{\xrightarrow{U}\}_{U \in \mathcal{U}}\rangle$ if Ob satisfies the following conditions. The symbol $\sqcup$ is the *quantum disjunction* $Y \sqcup Z = (Y^{\perp} \cap Z^{\perp})^{\perp}$.

1. If $S \in Ob$, then S is testable.
2. If $S \in Ob$, and for all testable subsets $Y \subseteq X$, if $S \cap Y \neq \emptyset$, then $S \subseteq Y$.
3. If $S \in Ob$, $T \in Ob$ and $S \neq T$, then for all $x \in S$ and $y \in T$, $x \perp y$.
4. $\bigsqcup_{Y \in Ob} Y = X$.

Condition 1 represents the testability of elements of bases because the elements of an orthonormal basis are one dimensional closed subspaces of a Hilbert space. Condition 2 represents the *atomicity* of a one-dimensional closed subspace of a Hilbert space. Thereby, one-dimensional closed subspaces are minimal closed subspaces of a Hilbert space except for $\emptyset$. Condition 3 means that all elements of an orthonormal basis are mutually orthogonal. Condition 4 represents the completeness of an orthonormal basis because quantum disjunction represents the closed subspace spanned by the elements in a Hilbert space.

For example, consider the following dynamic quantum frame $\langle X, \{\xrightarrow{Y?}\}_{Y \subseteq X}, \{\xrightarrow{U}\}_{U \in \mathcal{U}}\rangle$, which is an abstract representation of a part of two-dimensional Hilbert space H_2.

$X = \{x, y, z, w\}$.
$\xrightarrow{\{\emptyset\}?} = \emptyset$.
$\xrightarrow{\{x\}?} = \{(x, x), (z, x), (w, x)\}$.

$\xrightarrow{\{y\}?} = \{(y,y),(z,y),(w,y)\}$.
$\xrightarrow{\{z\}?} = \{(z,z),(x,z),(y,z)\}$.
$\xrightarrow{\{w\}?} = \{(w,w),(x,w),(y,w)\}$.
For all other sets $Y \subseteq X$, $\xrightarrow{Y?} = \{(x,x),(y,y),(z,z),(w,w)\}$.
$\xrightarrow{U_1} = \{(x,z),(z,y),(y,w),(w,x)\}$.
$\xrightarrow{U_2} = \{(x,y),(z,w),(y,x),(w,z)\}$.
For all other $V \in \mathcal{U}$, $\xrightarrow{V} = \{(x,x),(y,y),(z,z),(w,w)\}$.

Intuitively, $\{x,y,z,w\}$ corresponds to $\{|0\rangle, |1\rangle, (|0\rangle+|1\rangle)/\sqrt{2}, (|0\rangle-|1\rangle)/\sqrt{2}\}$ in H_2, U_1 corresponds to $\pi/4$ rotation, and U_2 corresponds to $\pi/2$ rotation. The following can be confirmed.

$x \perp y$ and $z \perp w$.
$\{\{x\},\{y\}\}$ and $\{\{z\},\{w\}\}$ are orthonormal bases.

To solve the second problem, we must introduce a new subset of propositional variables to represent an orthonormal basis. We define a set of *propositional variables for orthonormal basis* $Bp = \{s, t, \ldots\} \subset \{p, q, \ldots\}$, where both Bp and $\{p, q, \ldots\} - Bp$ are infinite sets. We introduce a new modal operator $\boxdot$ to solve the third problem. We regard this new modal operator as quantification of $[s?]$. Therefore, intuitively, $\boxdot A$ corresponds to $[s? \cup t? \cup \ldots]A$. This property is the same as the property of $\Box$, namely that $\Box$ is a quantification of $[p?]$.

We say that $\langle X, \{\xrightarrow{Y?}\}_{Y\subseteq X}, \{\xrightarrow{U}\}_{U\in\mathcal{U}}, Ob\rangle$ is an *extended dynamic quantum frame* if it satisfies the following conditions.

1. $\langle X, \{\xrightarrow{Y?}\}_{Y\subseteq X}, \{\xrightarrow{U}\}_{U\in\mathcal{U}}\rangle$ is a dynamic quantum frame.
2. Ob is an orthonormal basis of $\langle X, \{\xrightarrow{Y?}\}_{Y\subseteq X}, \{\xrightarrow{U}\}_{U\in\mathcal{U}}\rangle$.

We say that $\langle X, \{\xrightarrow{Y?}\}_{Y\subseteq X}, \{\xrightarrow{U}\}_{U\in\mathcal{U}}, Ob, V\rangle$ is an *extended dynamic quantum model* (EDQM) if it satisfies the following conditions.

1. $\langle X, \{\xrightarrow{Y?}\}_{Y\subseteq X}, \{\xrightarrow{U}\}_{U\in\mathcal{U}}, Ob\rangle$ is an extended dynamic quantum frame.
2. V is a function that assigns each propositional variable p (including $s \in Bp$) to a subset of X
3. For all $s \in Bp$, $V(s) \in Ob$.
4. For every $Y \in Ob$, there exists $s \in Bp$ such that $V(s) = Y$.

We define $\| \boxdot A\|$ in an EDQM as follows.

$\| \boxdot A\| = \{x \in X|$ for all $y \in X$ and for all $s \in Bp$, if $x(s?)y$, then $y \in \|A\|\}$

The truth and validity of the formulas for an EDQM are defined in the same way as for a DQM. From the conditions for an EDQM, every element of Ob has some $s \in Bp$, and each $V(s)$ can only be $Y \in Ob$. Therefore, if we regard Ob as the orthonormal basis for physical quantity M, then we can read $\boxdot A$ as "After a measurement of M, (whatever the value), A is true."

We add the following rules and axioms to **PDQL** and create a new logic that we refer to as **EPDQL** (extended propositional dynamic quantum logic). We have to restrict the substitution rule for validity because $V(s)(s \in Bp)$ can only be $Y \in Ob$.

(Test Generalization for Bp): If $A \to [s?]B$ is provable for all s, then infer $A \to \boxdot B$
(Testability Axiom for Bp): $\boxdot A \to [s?]A$
(Testability of Basis): $T(s)$
(Atomicity of Basis): $s \wedge A \wedge T(A) \to \Box\Box(s \to A)$
(Orthogonality of Bases): $s \to t \vee \sim t$
(Completeness of Orthonormal Basis): $\neg \boxdot \bot$
(Substitution Rule for **EPDQL**): If A is provable and $p \notin Bp$, then infer $A[p/B]$

Note that because Bp is a part of $\{p, q, \ldots\}$, axioms in **PDQL** that include $[p?]$ also apply to $[s?]$. We say that a rule is valid in an EDQM if it satisfies the following. If the premise of a rule is valid in an EDQM, then the conclusion of the rule is also valid in the EDQM.

Theorem 1. *All axioms and rules of* **EPDQL** *are valid in all EDQMs.*

Proof. The proofs of the validity of the generalization rule and testability axiom are the same as usual.

$T(s)$ is always true because of $V(s) \in Ob$.

Suppose $s \wedge A \wedge T(A)$ is true at $x \in X$. From $V(s) \in Ob$ and the second condition of an orthonormal basis, $s \to A$ is true at all $y \in X$. Therefore, $\Box\Box(s \to A)$ is true at x.

Suppose s is true at $x \in X$. As $V(s) \in Ob$ and $V(t) \in Ob$, if t is not true at x, $x \perp V(t)$. Therefore, $\Box\neg t$ is true at x.

From $\bigsqcup_{Y \in Ob} Y = X$, $\bigcap_{Y \in Ob} Y^{\perp} = (\bigsqcup_{Y \in Ob} Y)^{\perp} = \emptyset$. Therefore, each $x \in X$ is related by $\not\perp$ to some $y \in Y \in Ob$. From condition 4 of an EDQM and from Lemma 1, there exists $z \in X$ such that $x(s?)z$. Therefore, $\boxdot\bot$ cannot be true at any $x \in X$.

Suppose $s \wedge A$ is true at $x \in X$. For every t, $[t?]A$ is true at x because if $V(s) = V(t)$, $[t?]A = [s?]A$ and $s \wedge A \to [s?]A$ is valid in any EDQM. If $V(s) \neq V(t)$, because $x \perp V(t)$, $[t?]B$ is true for every B. Therefore, $\boxdot A$ is true at x. $\Box$

We can prove important formulas in **EPDQL**. For example, $s \wedge A \to \boxdot A$ (Eigenstate) and $\boxdot A \to \boxdot \boxdot A$ (Repeatability of measurement) can be proved as follows (we give only an outline of the proof):

The proof for $s \wedge A \to \boxdot A$.

1. From the partial functionality and the adequacy, $t \wedge A \to [t?]A$ (t does not appear in A).
2. From $\Box\neg t \to [t?]\bot$ ([7]) and $[t?]\bot \to [t?]A$, $\Box\neg t \wedge A \to [t?]A$.
3. From 1 and 2, $(t \vee \Box\neg t) \wedge A \to [t?]A$.
4. From the orthogonality of bases and 3, $s \wedge A \to [t?]A$ (s does not appear in A).
5. From the test generalization for Bp and 4, $s \wedge A \to \boxdot A$.

The proof for $\boxdot A \to \boxdot\boxdot A$.

1. $s \wedge A \to \boxdot A$ (s does not appear in A).
2. From the necessitation rule, the Kripke axiom and 1, $[s?]s \wedge [s?]A \to [s?]\boxdot A$.
3. From repeatability, the testability of s and 2, $[s?]A \to [s?]\boxdot A$.
4. $\boxdot A \to [s?]A$.
5. From 3 and 4, $\boxdot A \to [s?]\boxdot A$.
6. From the test generalization for Bp and 5, $\boxdot A \to \boxdot\boxdot A$.

Conversely, we can make an orthonormal basis of a dynamic quantum frame $\langle X, \{\xrightarrow{Y?}\}_{Y \subseteq X}, \{\xrightarrow{U}\}_{U \in \mathcal{U}}\rangle$ from the DQM $\langle X, \{\xrightarrow{Y?}\}_{Y \subseteq X}, \{\xrightarrow{U}\}_{U \in \mathcal{U}}, V\rangle$ by using these axioms. We use the definition of $\|\boxdot A\|$ not only in an EDQM but also in a DQM. We can do this because Bp is simply a part of $\{p, q, \ldots\}$.

Theorem 2 (Complete axiomatization for orthonormal basis). *If all axioms and rules of* **EPDQL** *are valid in a T-complete DQM* $\langle X, \{\xrightarrow{Y?}\}_{Y \subseteq X}, \{\xrightarrow{U}\}_{U \in \mathcal{U}}, V\rangle$, *then* $\{\|s\| | s \in Bp\}$ *is an orthonormal basis of* $\langle X, \{\xrightarrow{Y?}\}_{Y \subseteq X}, \{\xrightarrow{U}\}_{U \in \mathcal{U}}\rangle$.

Proof. For condition 1 of an orthonormal basis, the proof is straightforward from $T(s)$. For condition 2, the proof is straightforward from $s \wedge A \wedge T(A) \to \Box\Box(s \to A)$ for every A and the T-completeness of the model. For condition 3, suppose $x \in \|s\|$. From $s \to t \vee \sim t$, $x \in \|t\|$ or $x \in \|\sim t\|$. If $x \in \|t\|$, from conditions 1 and 2, $\|s\| = \|t\|$. If $x \in \|\sim t\|$, there is no $y \in X$ such that $x \not\perp y$ and $y \in \|t\|$. Therefore, for all $y \in \|t\|$, $x \perp y$. For condition 4, from $\neg\boxdot\bot$, condition 1 and the definition of $\|\boxdot A\|$, for any $x \in X$, there exist $s \in Bp$ and $y \in X$ such that $y \in \|s\|$ and $x(s?)y$. Therefore, all $x \in X$ are related to some $\|s\|(s \in Bp)$ by $\not\perp$. This implies that $\bigcap_{s \in Bp} \|s\|^{\perp} = \emptyset$. Therefore, $\bigsqcup \|s\| = (\bigcap_{s \in Bp} \|s\|^{\perp})^{\perp} = X$. $\Box$

Next, we consider multiple physical quantities. A Hilbert space has many of orthonormal bases. In general, each physical quantity corresponds to a different orthonormal basis. In a Hilbert space, we can generate another orthonormal basis from a given one by applying unitary transformations. Therefore, if $\{a_1, a_2, \ldots\}$ is an orthonormal basis of a Hilbert space and if $U \in \mathcal{U}$, then $\{Ua_1, Ua_2, \ldots\}$ is also an orthonormal basis of the Hilbert space. We introduce this notion into the EDQM. We define $x_U \in X$ and sets $Y_U \subseteq X$ and Ob_U below.

If $x(U)y$, then $x_U = y$

$Y_U = \{x \in X | \exists y \in Y \text{ and } y(U)x\}$
$Ob_U = \{Y_U \subseteq X | Y \in Ob\}$

Then, we can prove the following lemma and theorem for the EDQM in the same way as for a Hilbert space.

Lemma 2. *In any dynamic quantum frame* $\langle X, \{\xrightarrow{Y?}\}_{Y \subseteq X}, \{\xrightarrow{U}\}_{U \in \mathcal{U}}\rangle$*, for all* $x, y \in X$*, if* $x \perp y$*,* $x(U)z$ *and* $y(U)w$*, then* $z \perp w$*. Furthermore, if* $x \not\perp y$*,* $x(U)z$ *and* $y(U)w$*, then* $z \not\perp w$*.*

Proof. For the sake of contradiction, suppose $x \perp y$, $x(U)z$, $y(U)w$ and $z \not\perp w$. Because U is adjoint, there exists $v \in X$ such that $w(U^\dagger)v$ and $v \not\perp x$. However, this contradicts $w(U^\dagger)y$ and the unitary functionality of $U^\dagger$.

The proof for the case of $x \not\perp y$ is almost the same as the above proof. □

From this lemma, we can prove that $(Y^\perp)_U = {Y_U}^\perp$.

Theorem 3. *If* Ob *is an orthonormal basis of a dynamic quantum frame* $\langle X, \{\xrightarrow{Y?}\}_{Y \subseteq X}, \{\xrightarrow{U}\}_{U \in \mathcal{U}}\rangle$*, then* Ob_U *is also an orthonormal basis of* $\langle X, \{\xrightarrow{Y?}\}_{Y \subseteq X}, \{\xrightarrow{U}\}_{U \in \mathcal{U}}\rangle$*.*

Proof. For condition 1 of an orthonormal basis, for the sake of contradiction, suppose Y is testable but Y_U is not testable. Then, there exists $x \in X$ such that $x \perp {Y_U}^\perp$ and $x \notin Y_U$. From Lemma 2 and the unitary functionality of U and $U^\dagger$, $x_{U^\dagger} \perp Y^\perp$ and $x_{U^\dagger} \notin Y$. This is a contradiction. Therefore, for all $Y \in Ob_U$, Y is testable.

For condition 2, for the sake of contradiction, suppose $Y \in Ob_U$, $Z \subset Y$, $Z \neq Y$, $Z \neq \emptyset$ and that Z is testable. From Lemma 2 and the above argument, we have $Y_{U^\dagger} \in Ob$, $Z_{U^\dagger} \subset Y_{U^\dagger}$, $Z_{U^\dagger} \neq Y_{U^\dagger}$, $Z_{U^\dagger} \neq \emptyset$, and that $Z_{U^\dagger}$ is testable. This contradicts the atomicity of the elements of Ob.

For condition 3, from Lemma 2, if $Y \perp Z$, then $Y_U \perp Z_U$.

For condition 4, for the sake of contradiction, suppose $\bigcap_{Y \in Ob_U} Y^\perp \neq \emptyset$. Then, there exists $x \in X$ such that for all $y \in Y \in Ob_U$, $x \perp y$. From Lemma 2, for all $z \in Z \in Ob$, $x_{U^\dagger} \perp z$. This contradicts the completeness of Ob. □

From Theorem 3, the conditions for $V(s)$, and unitary functionality, we can prove that for all $s \in Bp$, $\|[U^\dagger]s\| \in Ob_U$ in an EDQM. Furthermore, for all $Y \in Ob_U$, there exists $t \in Bp$ such that $\|[U^\dagger]t\| = Y$. Therefore, we now introduce new modal symbols $\boxdot_U$ for Ob_U. Because we have multiple symbols for unitary evolution, we introduce as many symbols $\boxdot_U$ as there are U. We define $\| \boxdot_U A\|$ in an EDQM as follows.

$\| \boxdot_U A\| = \{x \in X| \text{ for all } s \in Bp, \text{ if } x(([U^\dagger]s)?)y, \text{ then } y \in \|A\|\}$

We add the following axioms to **EPDQL** and obtain a new axiom system that we refer to as **EPDQLU** (extended propositional dynamic quantum logic with unitary transformations of bases).

(Test Generalization for $U^\dagger(Bp)$): If $A \to [([U^\dagger]s)?]B$ is provable for all s, then infer $A \to \boxdot_U B$
(Testability Axiom for $U^\dagger(Bp)$): $\boxdot_U A \to [([U^\dagger]s)?]A$
(Testability of Basis): $T([U^\dagger]s)$
(Atomicity of Basis): $[U^\dagger]s \wedge A \wedge T(A) \to \Box\Box([U^\dagger]s \to A)$
(Orthogonality of Bases): $[U^\dagger]s \to [U^\dagger]t \vee \sim [U^\dagger]t$
(Completeness of Orthonormal Basis): $\neg \boxdot_U \bot$

We define these axioms simply by changing $s \in Bp$ in the axioms of **EPDQL** to $[U^\dagger]s$, and by changing $\boxdot$ to $\boxdot_U$. Note that some of these axioms are provable in **EPDQL** if an axiom does not include $\boxdot_U$.

Theorem 4. *All axioms and rules of* **EPDQLU** *are valid in all EDQMs.*

Proof. From the property of Ob_U and the definition of $\| \boxdot_U A\|$, we can trace the proof of Theorem 1. $\Box$

Theorem 5 (Complete axiomatization for orthonormal basis Ob_U). *If all axioms of* **EPDQLU** *are valid in a T-complete DQM $\langle X, \{\xrightarrow{Y?}\}_{Y \subseteq X}, \{\xrightarrow{U}\}_{U \in \mathcal{U}}, V\rangle$, then $\{\|[U^\dagger]s\| | s \in Bp\}$ is an orthonormal basis of $\langle X, \{\xrightarrow{Y?}\}_{Y \subseteq X}, \{\xrightarrow{U}\}_{U \in \mathcal{U}}\rangle$.*

Proof. We can trace the proof of Theorem 2. $\Box$

Another important notion for quantum physics is *mutually unbiased bases* in a Hilbert space. Suppose $a = \{a_1, a_2, ..., a_d\}$ and $b = \{\{b_1, b_2, ..., b_d\}$ are orthonormal bases of a d-dimensional Hilbert space H. We say these two orthonormal bases are mutually unbiased bases if $|(a_i, b_j)|^2 = 1/d$ for all $i, j \in \{1, 2, ..., d\}$ where (a, b) is the inner product of a and b. Intuitively, this concept represents that every element of the orthonormal basis equally contains the elements of the other orthonormal basis. We use this concept to describe some physical quantities under the uncertainty principle. For example, $\{|0\rangle, |1\rangle\}$ and $\{(|0\rangle + |1\rangle)/\sqrt{2}, (|0\rangle - |1\rangle)/\sqrt{2}\}$ in H_2 are mutually unbiased bases. The situation is a little different because it is a concept in an infinite dimensional Hilbert space H, but an orthonormal basis for position and an orthonormal basis for momentum are mutually unbiased bases in H. However, in a dynamic quantum frame, we can only determine whether states x and y are orthogonal or not. In other words, we cannot represent the degree of non-orthogonality in a dynamic quantum frame. Therefore, we abstract the notion of mutually unbiased bases and define *quasi-mutually unbiased bases* in a dynamic quantum frame. We only care if an element contains all elements of the other orthonormal basis or not (that is, orthogonal or not). We say that orthonormal bases Ob and Ob' are quasi-mutually unbiased bases of a dynamic quantum frame $\langle X, \{\xrightarrow{Y?}\}_{Y \subseteq X}, \{\xrightarrow{U}\}_{U \in \mathcal{U}}\rangle$ if Ob and Ob' satisfy the following conditions.

1. For all $y \in Y \in Ob'$ and $Z \in Ob$, there exists $z \in Z$ such that $y \not\perp z$.
2. For all $y \in Y \in Ob$ and $Z \in Ob'$, there exists $z \in Z$ such that $y \not\perp z$.

We can use the following axioms for quasi-mutually unbiased bases of U.

$T(A) \wedge \neg \boxdot_U \neg \boxdot A \to \Box\Box A$
$T(A) \wedge \neg \boxdot \neg \boxdot_U A \to \Box\Box A$

Theorem 6. *In an EDQM* $\langle X, \{\xrightarrow{Y?}\}_{Y\subseteq X}, \{\xrightarrow{U}\}_{U\in\mathcal{U}}, Ob, V\rangle$, *if* Ob *and* Ob_U *are quasi-mutually unbiased bases of* $\langle X, \{\xrightarrow{Y?}\}_{Y\subseteq X}, \{\xrightarrow{U}\}_{U\in\mathcal{U}}\rangle$, *then the axioms for quasi-mutually unbiased bases of* U *are valid in* $\langle X, \{\xrightarrow{Y?}\}_{Y\subseteq X}, \{\xrightarrow{U}\}_{U\in\mathcal{U}}, Ob, V\rangle$.

Proof. Suppose $T(A)$ and $\neg \boxdot_U \neg \boxdot A$ are true at $x \in X$. Then, there exist $y \in X$ and s such that $x([U^\dagger]s?)y$ and $y \in \| \boxdot A\|$. Because $\|[U^\dagger]s\|$ is testable, $[U^\dagger]s$ is true at y by repeatability. Therefore, there exists $Y \in Ob_U$ such that $y \in Y$. From Lemma 1 and the assumption that Ob and Ob_U are quasi-mutually unbiased bases, for all $Z \in Ob$, there exist $t \in Bp$ and $z \in Z$ such that $\|t\| = Z$ and $y(t?)z$. From $y \in \| \boxdot A\|$, A is true at z. From $T(A)$ and atomicity, A is true at all $z' \in Z$. Therefore, A is true at all $z \in Z \in Ob$. Finally, from the completeness of Ob and the testability of $\|A\|$, $\|A\| = X$. Therefore, $\Box\Box A$ is true at x.

The proof for $T(A) \wedge \neg \boxdot \neg \boxdot_U A \to \Box\Box A$ is almost the same as this proof. □

Theorem 7. *In a T-complete DQM* $\langle X, \{\xrightarrow{Y?}\}_{Y\subseteq X}, \{\xrightarrow{U}\}_{U\in\mathcal{U}}, V\rangle$, *if* $\{\|s\| | s \in Bp\}$ *and* $\{\|[U^\dagger]s\| | s \in Bp\}$ *are orthonormal bases, and if the axioms for quasi-mutually unbiased bases of* U *are valid, then* $\{\|s\| | s \in Bp\}$ *and* $\{\|[U^\dagger]s\| | s \in Bp\}$ *are quasi-mutually unbiased bases of* $\langle X, \{\xrightarrow{Y?}\}_{Y\subseteq X}, \{\xrightarrow{U}\}_{U\in\mathcal{U}}\rangle$.

Proof. For the sake of contradiction, suppose $y \in Y \in \{\|s\| | s \in Bp\}$ but that there is no $z \in \|[U^\dagger]t\|$ such that $y \not\perp z$. From the T-completeness of the model and the testability of $\sqcup$, there exists A such that $\bigsqcup\{\{\|[U^\dagger]s\| | s \in Bp\} - \|[U^\dagger]t\|\} = \|A\|$. Because $y \perp \|[U^\dagger]t\|$, $\boxdot_U A$ is true at y. From $y \in Y \in \{\|s\| | s \in Bp\}$, $\neg \boxdot \neg \boxdot_U A$ is true at y. From the nature of an orthonormal basis, $\|[U^\dagger]t\| \cap \|A\| = \emptyset$ and $[U^\dagger]t \neq \emptyset$. Therefore, there exists $x \in X$ such that $x \notin \|A\|$. This contradicts the validity of the axiom because $T(A)$ and $\neg\boxdot\neg\boxdot_U A$ are true at y but $\Box\Box A$ is not true at y.

The proof for the opposite direction is almost the same as this proof by using $T(A) \wedge \neg \boxdot_U \neg \boxdot A \to \Box\Box A$. □

4 Conclusions and Remarks

We constructed a model for DQL with an orthonormal basis. We also introduced new modal symbols and axioms for these symbols. These modal symbols bring a new type of proposition to DQL, and these propositions are essential in quantum physics.

We used a T-complete model in this study to express every testable subset of X by formulas to establish some theorems. Another way to deal with testable subsets is using formulas in some definitions. For example, we can change condition 2 of an orthonormal basis as follows:

If $S \in Ob$, and for all testable sets $\|A\|$, if $S \cap \|A\| \neq \emptyset$, then $S \subseteq \|A\|$.

If we use this definition, an orthonormal basis is defined for a DQM rather than for a dynamic quantum frame. Under this definition, we do not have to assume T-completeness because we need only deal with subsets of X that are expressed by $\|A\|$ for some A. However, if we use this definition, the discussion becomes complicated in many ways. For example, as $V(s)$ is already defined by a DQM, we have to use complicated definitions to define the EDQM.

Based on the results of this study, if we develop *many valued quantum logic*, we may be able to use mutually unbiased bases completely because many valued quantum logic includes the notion of the degree of non-orthogonality. [25] can be cited as one of the studies of many valued quantum logic. The degree of orthogonality is expressed by noting a number on the modal symbol for orthogonality. However, to express the properties of a Hilbert space in more detail, more conditions are required than the conditions introduced in [25]. Therefore, to combine this study with the modal symbol that introduced the numerical value, a more detailed analysis of the modal symbol with the numerical value is required.

Whether all DQMs have at least one orthonormal basis is debatable because some DQMs might have no atomic states. If we use models that are created from a concrete Hilbert space using the method in [2], these models obviously have orthonormal bases.

[26] and [27] can be cited as studies that analyzed the orthogonal relation in another way. Intuitively, in these studies, the orthogonal relation is analyzed in a more abstract direction based on *ortho logic* (OL) rather than DQL. Some conditions related to orthogonality are added to OL and its nature is analyzed. Although the direction is slightly different from the analysis of the orthonormal basis, it may be possible to analyze the orthogonality in more detail while maintaining the degree of abstraction by combining it with the analysis of the model constructed in this study.

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References

1. Baltag, A., Smets, S.: Correlated Knowledge: an Epistemic-Logic View on Quantum Entanglement. International Journal of Theoretical Physics. 49(12), 3005–3021 (2010)
2. Baltag, A., Smets, S.: The logic of quantum programs. QPL 2004. 39–56 (2004)
3. Baltag, A., Smets, S.: A Dynamic-Logical Perspective on Quantum Behavior. Studia Logica. 89, 187-—211 (2008)
4. Baltag, A., Smets, S.: Modeling correlated information change: from conditional beliefs to quantum conditionals. Soft Computing 21(6), 1523—1535 (2017)
5. Baltag, A., Smets, S.: The Dynamic Logic of Quantum Information. Mathematical Structures in Computer Science. 16(3), 491–525 (2006)

6. Baltag, A., Smets, S.: Quantum logic as a dynamic logic. Synthese. 179, 285–306 (2011)
7. Baltag, A., Smets, S.: The dynamic turn in quantum logic. Synthese. 186(3), 753–773 (2012)
8. Beltrametti, E., Dalla Chiara M. L., Giuntini, R, Leporini, R., Sergioli, G.: A Quantum Computational Semantics for Epistemic Logical Operators. Part I: Epistemic Structures. International Journal of Theoretical Physics. 53(10), 3279–3292 (2013)
9. Beltrametti, E., Dalla Chiara M. L., Giuntini, R, Leporini, R., Sergioli, G.: A Quantum Computational Semantics for Epistemic Logical Operators. Part II: Semantics. International Journal of Theoretical Physics. 53(10), 3293–3307 (2013)
10. Birkhoff, G., Von Neumann, J.: The Logic of Quantum Mechanics. The Annals of Mathematics. 37(4), 823–843 (1936)
11. Chiara, M. L. D., Giuntini, R.: Quantum Logics. Gabbay, D. M., Guenthner, F. (ed.): Handbook Of Philosophical Logic 2nd Edition. 6 (1), 129–228 (2002)
12. Engesser, K., Gabbay, D., Lehmann, D.: Handbook of Quantum Logic and Quantum Structures: Quantum Logic. Elsevier Science B.V., Amsterdam (2009)
13. Fischer, M, J., Ladner, R, E.: Propositional dynamic logic of regular programs. Journal of Computer and System Sciences. 18(2), 194–211 (1979)
14. Fritz, T.: Quantum logic is undecidable. arXiv:1607.05870. (2016)
15. Harel, D., Kozen, D., Tiuryn, J.: Dynamic Logic. The MIT press. (2000)
16. Jauch, J.: Foundations of Quantum Mechanics. Addison-Wesley Publishing Company, Reading (1968)
17. Negri, S.: Proof Analysis in Modal Logic. Journal of Philosophical Logic. 34, 507–544 (2005)
18. Negri, S.: Proof theory for modal logic. Philosophy Compass. 6(8), 523–538 (2011)
19. Nielsen, M, A., Isaac, L, C.; Quantum Computation and Quantum Information. Cambridge University Press. (2000)
20. Nishimura, H.: Sequential Method in Quantum Logic. The Journal of Symbolic Logic. 45(2), 339–352 (1980)
21. Nishimura, H.: Proof Theory for Minimal Quantum Logic I. International Journal of Theoretical Physics. 33(1), 103–113 (1994)
22. Nishimura, H.: Proof Theory for Minimal Quantum Logic II. International Journal of Theoretical Physics. 33(7), 1427–1443 (1994)
23. Piron, C.: Foundations of Quantum Physics. W.A. Benjamin Inc., Reading (1976)
24. Rdei, M.: Quantum Logic in Algebraic Approach. Fundamental Theories of Physics. 91 (1998)
25. Tokuo, K.: Extended Quantum Logic. Journal of Philosophical Logic. 32, 549-563 (2003)
26. Zhong, S.: Correspondence between Kripke frames and projective geometries. Studia Logica 106, 167–189 (2018)
27. Zhong, S.: On the Modal Logic of the Non-orthogonality Relation Between Quantum States. Journal of Logic, Language and Information. 27(2), 157—-173 (2018)

Cautious distributed belief

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Abstract This manuscript introduces and studies a notion of *cautious* distributed belief. Different from the standard distributed belief, the cautious distributed belief of a group is inconsistent only when *all* group members are individually inconsistent. The paper presents basic results about cautious distributed belief, investigates whether it inherits properties from individual belief, and compares it with standard distributed belief. Although both notions are equivalent in the class of reflexive models, this is not the case in general. The propositional language extended only with cautious distributed belief is strictly less expressive than the propositional language extended only with standard distributed belief. We, finally, identify a minimal extension of the language making the former as expressive as the latter.

Keywords: cautious distributed belief · distributed belief · epistemic logic · expressivity · bisimilarity

1 Introduction

Epistemic logic (*EL*; [12]) is a simple and yet powerful framework for representing the knowledge of a set of agents. Semantically, it typically relies on relational 'Kripke' models, assigning to each agent a binary *indistinguishability* relation over possible worlds (i.e., possible states of affairs). Syntactically, it uses the agent's indistinguishability to define her *knowledge*: at a world w an agent i knows that φ is the case if and only if φ holds in all the situations that are, for her, indistinguishable from w. Despite its simplicity, *EL* has become a wide-spread tool, contributing to the formal study of complex multi-agent epistemic phenomena in philosophy [9], computer science [6,14] and economics [4,15].

One of the most attractive features of *EL* is that one can reason not only about individual knowledge, but also about different forms of knowledge for groups. A historically important example is the notion of *common knowledge* [13], which is known to be crucial in social interactions.[1] Another important epistemic notion for groups, key in distributed systems, is that of *distributed knowledge* [11,7,8]. Intuitively, a group has distributed knowledge of φ if and only if φ follows from the combination of the individual knowledge of all its members. In *EL* (which, recall, uses uncertainty to define knowledge), this intuition has a

[1] A group has common knowledge of φ if and only if everybody in the group knows φ, everybody in the group knows that everybody in the group knows φ, and so on.

natural representation: at a world w a group G has distributed knowledge of φ if and only if φ holds in all the situations that *all the members of the group* consider indistinguishable (i.e., if and only if φ holds in all the situations *no one in the group* can distinguish) from w. In other words, the indistinguishability relation for the distributed knowledge of a group G corresponds to the *intersection* of the indistinguishability relation of G's members.

Since distributed knowledge is the result of combining the individual knowledge of different agents, one can wonder whether agents might have inconsistent distributed knowledge (i.e., whether it is possible for a set of agents to know $\perp$ distributively). When one works with a *truthful* notion of knowledge (semantically, when all indistinguishability relations are required to be reflexive), distributed knowledge does not have this problem: all indistinguishability relations contain the reflexive edges, and thus their intersection will never be empty. However, when one works with weaker notions of information, counterintuitive situations might occur. For example, if one works with a notion of beliefs (typically represented by using a serial, transitive and Euclidean relation; see, e.g., [12]), it is possible for all agents to be consistent (i.e., no one of them believes contradictions), and yet their distributed beliefs might contain $\perp$.

This paper introduces and studies a notion of *cautious* distributed belief (modality: $D^{\forall}$). It has the property that it does not become inconsistent in the case of mutual inconsistency, picking instead a form of maximally consistent combined information. The intuition behind it is that, although a group G as a whole might be inconsistent at some world w (i.e., the set of worlds *everybody* in G considers possible from w is empty), there might be consistent subgroups among which the maximal ones become important. Considering notions of maximal consistency is a standard approach in non-monotonic reasoning for resolving potential conflicts.[2] As its name suggest, $D^{\forall}$ uses these maximally consistent subgroups of agents in a cautious way: at a world w a group G has cautious distributed belief that φ if and only if *every* maximally consistent subgroup of G has distributed belief that φ.[3]

The manuscript is organised as follows. Section 2 recalls the definition of a relational 'Kripke' model as well as that of the standard distributed belief operator D. Then it introduces the notion of cautious distributed belief, using a relatively simple example to compare the two notions, and presenting some basic results about it. Section 3 studies whether this notion of belief for groups inherits properties from the individual beliefs of the group's members. Section 4 compares the expressive power of both modalities, showing that a modal language with only $D^{\forall}$ is strictly less expressive than a modal language with only D; it does so

[2] Think, e.g., about the *extensions* of a theory in default logic [16], or the *maximally admissible* (i.e., preferred) sets of arguments in abstract argumentation theory [5]. The idea has been also used within epistemic logic (e.g., by [2] in the context of evidence-based beliefs) and also for distributed beliefs (by [10], in the context of *explicit* beliefs defined via belief bases).

[3] This corresponds to the *skeptical* reasoner in non-monotonic reasoning. There is also an alternative that matches the *credulous* reasoner, discussed briefly in Section 5.

by providing a notion of bisimulation for $D^{\forall}$. Yet, the paper identifies what it is that D can see but $D^{\forall}$ cannot. Finally, Section 5 summarises the results and discusses further research lines.

2 Basic definitions

Throughout this text, let A be a finite non-empty set of agents and P be a countable non-empty set of atomic propositions. The basic propositional language (using $\neg$ and $\wedge$ as primitive operators) is denoted by $\mathcal{L}$. (Its semantic interpretation is as usual.) Then, $\mathcal{L}_{X_1,\ldots,X_n}$ is the language extending $\mathcal{L}$ with the operators $X_1, \ldots, X_n$. In particular, $\mathcal{L}_D$ is $\mathcal{L}$ with the additional use of D_G for $\varnothing \neq G \subseteq A$, and $\mathcal{L}_{D^{\forall}}$ is $\mathcal{L}$ with the additional use of $D_G^{\forall}$ for $\varnothing \neq G \subseteq A$.

Definition 1 (Belief model) *A* belief model *is a tuple $\mathcal{M} = \langle W, R, v\rangle$ where W is a non-empty set of possible worlds (also denoted as $\mathrm{D}(\mathcal{M})$), $R = \{R_a \subseteq W \times W \mid a \in A\}$ assigns an arbitrary* accessibility *relation to each agent $a \in A$, and $v : P \to 2^S$ is a valuation function. A* pointed belief model *is a pair $(\mathcal{M}, s)$ with $\mathcal{M}$ a belief model and $s \in \mathrm{D}(\mathcal{M})$ a world in it. The class of all belief models is denoted as $\mathbf{M}$. Given $\langle W, R, v\rangle$ in $\mathbf{M}$, $a \in A$ and $s \in W$, the set $C_a(s) := \{s' \in W \mid sR_as'\}$ is called* a's conjecture set at s. *The generalisation to a set of agents $G \subseteq A$, called* G's (combined) conjecture set at s, *is defined as $C_G(s) := \bigcap_{a\in G} C_a(s)$.* ◀

Belief models are nothing but multi-agent Kripke (relational) models. Thus, they allow us to represent not only the beliefs each individual agent has, but also different belief notions for groups. As discussed in the introduction, the focus here is the novel notion of *cautious distributed beliefs* ($D^{\forall}$), together with its relationship with the well-known notion of *distributed beliefs* (D). For the semantic interpretation of the first, the following definitions will be useful.

Definition 2 (Consistency and maximal consistency) *Let $\langle W, R, v\rangle$ be in $\mathbf{M}$. Take sets of agents $\varnothing \subset G' \subseteq G \subseteq A$ and a world $s \in W$. The set G' is* consistent at s *if and only if $C_{G'}(s) \neq \varnothing$. It is* maximally consistent at s w.r.t. G *(notation: $G' \subseteq_s^{max} G$) if and only if it is consistent at s and, additionally, every H satisfying $G' \subset H \subseteq G$ is inconsistent (i.e., $C_H(s) = \varnothing$). Finally, the set $C_G^{\forall}(s) := \bigcup_{G' \subseteq_s^{max} G} C_{G'}(s)$ (the* consistent (combined) conjecture set *of G at s) contains the worlds that are relevant for the maximally consistent subgroups of G at world s. The* cautious distributed belief relation $R_G^{\forall} \subseteq \mathrm{D}(\mathcal{M}) \times \mathrm{D}(\mathcal{M})$, *given by $sR_G^{\forall}t$ iff $t \in C_G^{\forall}(s)$, will simplify some later work.* ◀

Here is the semantic interpretation of the two operators, D and $D^{\forall}$, together with the standard operator for individual belief B. We also present the semantics of an additional constant $\asymp_G$, which will be useful later. Languages using these operators will be discussed in Section 4.

Definition 3 (Two types of distributed belief) *Let $(\mathcal{M}, s)$ be a pointed belief model with $\mathcal{M} = \langle W, R, v\rangle$; take $a \in A$ and a non-empty $G \subseteq A$. Then,*

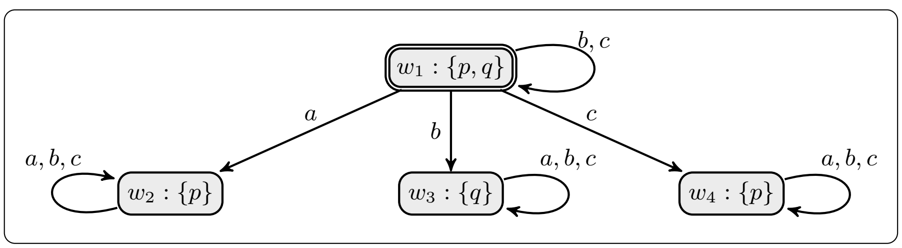
$$\begin{array}{lll}
\mathcal{M}, s \vDash B_a\,\varphi & \textit{iff} & \forall s' \in C_a(s)\text{: } \mathcal{M}, s' \vDash \varphi, \\
\mathcal{M}, s \vDash D_G\,\varphi & \textit{iff} & \forall s' \in C_G(s)\text{: } \mathcal{M}, s' \vDash \varphi, \\
\mathcal{M}, s \vDash D^{\forall}_G\,\varphi & \textit{iff} & \forall G' \subseteq^{max}_s G,\ \forall s' \in C_{G'}(s)\text{: } \mathcal{M}, s' \vDash \varphi \\
 & & \textit{(equivalently, } \forall s' \textit{ such that } sR^{\forall}_G s'\text{: } \mathcal{M}, s' \vDash \varphi\textit{)},^{[4]} \\
\mathcal{M}, s \vDash \asymp_G & \textit{iff} & C_G(s) = \varnothing.
\end{array}$$

A formula φ *is* valid in a class of belief models $\mathbf{C}$ *(notation:* $\mathbf{C} \vDash \varphi$*) when* φ *is true in every world of every model in* $\mathbf{C}$*. A formula is* valid *(notation:* $\vDash \varphi$*) when* $\mathbf{M} \vDash \varphi$*.* ◀

Note the difference between D_G and $D^{\forall}_G$. On the one hand, $D_G\,\varphi$ holds at s when every world in the conjecture set of G satisfies φ.[5] On the other hand, $D^{\forall}_G\,\varphi$ holds at s when every world in the conjecture set of *every maximally consistent subgroup of* G satisfies φ. In other words, $D^{\forall}_G\,\varphi$ holds at s if and only if *every* maximally consistent subgroup of G has distributed belief of φ. Note also how $\asymp_G$ simply expresses the fact that the conjecture set of G is inconsistent.

Here is an simple example showing the differences between D and $D^{\forall}$.

Example 1 *Consider the belief model* $\mathcal{M}$ *below.*[6] *Note how, at* w_1*,* a *believes* p *to be true and* q *to be false (*$\mathcal{M}, w_1, \vDash B_a\,p \land B_a\,\neg q$*). Nevertheless,* b *is uncertain about* p *but believes* q *to be true (*$\mathcal{M}, w_1 \vDash (\neg\, B_b\,p \land \neg\, B_b\,\neg p) \land B_b\,q$*). Finally,* c *believes* p *but is uncertain about* q *(i.e.,* $\mathcal{M}, w_1, \vDash B_c\,p \land (\neg\, B_c\,q \land \neg\, B_c\,\neg q)$*).*



Consider first the group $G_1 = \{a, b\}$*. On the one hand, both members of* G_1 *are individually consistent at* w_1 *and yet* $C_{G_1}(w_1) = \varnothing$*; thus, at* w_1*, the maximally consistent subgroups are* $\{a\}$ *and* $\{b\}$*. Their conjecture sets are* $C_a(w_1) = \{w_2\}$ *and* $C_b(w_1) = \{w_1, w_3\}$*, and hence* G_1*'s consistent conjecture set is* $C^{\forall}_{G_1}(w_1) = \{w_1, w_2, w_3\}$*. Thus,* $\mathcal{M}, w_1 \vDash \neg\, D^{\forall}_{G_1}\,p \land \neg\, D^{\forall}_{G_1}\,q$*. On the other hand, when we consider standard distributed belief, we see that* $\mathcal{M}, w_1 \vDash D_{G_1}\,p \land D_{G_1}\,q$*. This is*

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[4] The two definitions are equivalent. The first makes explicit the two quantification steps; the second, given in terms of the group's cautious distributed belief relation, reveals that $D^{\forall}_G$ is in fact a normal modality.

[5] In particular, individual belief operators B_a can be defined in terms of D, as $D_{\{a\}}\,\varphi$ (abbreviated as $D_a\,\varphi$) holds in a world s if and only if $\mathcal{M}, s' \vDash \varphi$ for all $s' \in C_a(s)$.

[6] Note: the individual relations are serial, transitive and Euclidean. While the paper uses the term "belief" in a rather loose way, these three properties are the ones commonly associated to a belief operator.

however due to the fact that $C_{G_1}(w_1) = \varnothing$ *and we end up quantifying over an empty set. Thus, we also get* $\mathcal{M}, w_1 \vDash D_{G_1} \perp$.

Now c *joins the group,* $G_2 = \{a, b, c\}$. *On the one hand, at* w_1 *both* b *and* c *are consistent (i.e., they can 'consistently combine information'); still,* a *and* c *are not. Thus, the maximally consistent sets are* $\{a\}$ *and* $\{b, c\}$. *The relevant conjecture sets are now* $C_a(w_1) = \{w_2\}$ *and* $C_{\{b,c\}}(w_1) = \{w_1\}$, *so* $C^{\forall}_{G_2}(w_1) = \{w_1, w_2\}$. *Then,* $\mathcal{M}, w_1 \vDash D^{\forall}_{G_2}\, p \wedge \neg D^{\forall}_{G_2}\, q$ *(the latter because, even though* b *and* c *together believe* q, *agent* a *remains 'a loner' and still believes that* q *is false). On the other hand, the situation with standard distributed belief remains as for* G_1*:* $\mathcal{M}, w_1 \vDash D_{G_2}\, p \wedge D_{G_2}\, q \wedge D_{G_2} \perp$. ◀

Some basic results about $D^{\forall}_G$. The standard notion of distributed belief, D_G, can be inconsistent even when every agent in G is consistent. The first result here shows that this is not the case for cautious distributed belief: it is inconsistent if and only if *all* agents in G are inconsistent.

Proposition 1 *For every non-empty* $G \subseteq A$ *we have* $\vDash D^{\forall}_G \perp \leftrightarrow \bigwedge_{a \in G} B_a \perp$.

Proof. Take any $\mathcal{M}$, *any* $s \in \mathrm{D}(\mathcal{M})$ *and any non-empty* $G \subseteq A$. *(⇒) If* $\mathcal{M}, s \vDash D^{\forall}_G \perp$ *then, because no world satisfies* $\perp$, *either* $C_{G'}(s) = \varnothing$ *for all* $G' \subseteq^{max}_s G$, *or there is no* G' *satisfying* $G' \subseteq^{max}_s G$. *But, by definition, no* G' *satisfying* $G' \subseteq^{max}_s G$ *is s.t.* $C_{G'}(s) = \varnothing$. *Hence, there is no* G' *satisfying* $G' \subseteq^{max}_s G$, *which means every* $G' \subseteq G$ *is s.t.* $C_{G'}(s) = \varnothing$. *In particular, all singletons* $\{a\}$ *for* $a \in G$ *are s.t.* $C_a(s) = \varnothing$, *and thus* $\mathcal{M}, s \vDash \bigwedge_{a \in G} B_a \perp$. *(⇐) If* $\mathcal{M}, s \vDash \bigwedge_{a \in G} B_a \perp$ *then* $C_a(s) = \varnothing$ *for every* $a \in G$. *Hence, every non-empty* $G' \subseteq G$ *is s.t.* $C_{G'}(s) = \varnothing$, *so there is no* G' *satisfying* $G' \subseteq^{max}_s G$. *Thus,* $\mathcal{M}, s \vDash D^{\forall}_G \perp$.■

For another basic result, recall that individual belief operators (B_a for $a \in A$) can be expressed using the distributed belief operator for singleton groups (D_a). The same can be done with cautious distributed belief. For any world s and any agent a, there is at most one maximally consistent subgroup of $\{a\}$, namely $\{a\}$ itself. Then, $C_a(s) = C^{\forall}_a(s)$ and hence agent a's individual belief and $\{a\}$'s cautious distributed belief coincide.

Proposition 2 $\vDash B_a\varphi \leftrightarrow D^{\forall}_{\{a\}}\, \varphi$. ■

Finally, an important property of standard distributed belief is *coalition monotonicity*: if a group $H \subseteq A$ has standard distributed belief that φ, then so does any extension $G \supseteq H$ (thus, $H \subseteq G \subseteq A$ implies $\vDash D_H\, \varphi \to D_G\, \varphi$). This is not the case for cautious distributed belief. This is because the agents that join the group might not be consistent with any of the ones that were there before. In such cases, when consistent, they will be part of a different maximally consistent subgroup, which might not have the distributed belief φ. This is shown in Example 1, where $\mathcal{M}, w_1 \vDash D^{\forall}_{\{b\}}\, q$ and yet $\mathcal{M}, w_1 \nvDash D^{\forall}_{\{a,b\}}\, q$. Thus,

Fact 1 $\nvDash D^{\forall}_H\, \varphi \to D^{\forall}_G\, \varphi$ *for* $H \subseteq G \subseteq A$. ■

Frame condition	Characterising formula
seriality (l): $\forall s \in W \; \exists t \in W \,.\, sSt$	*consistency*: $\Box\varphi \to \Diamond\varphi$
reflexivity (r): $\forall s \in W \,.\, sSs$	*truthfulness of knowledge/belief*: $\Box\varphi \to \varphi$
transitivity (t): $\forall s, t, u \in W \,.\, ((sSt \;\&\; tSu) \Rightarrow sSu)$	*positive introspection*: $\Box\varphi \to \Box\Box\varphi$
symmetry (s): $\forall s, t \in W \,.\, (sSt \Rightarrow tSs)$	*truthfulness of possible knowledge/belief*: $\Diamond\Box\varphi \to \varphi$
Euclidicity (e): $\forall s, t, u \in W \,.\, ((sSt \;\&\; sSu) \Rightarrow tSu)$	*negative introspection*: $\neg\Box\varphi \to \Box\neg\Box\varphi$

Table 1: Relational properties and their well-known characterising formula.

3 Inheriting relational properties

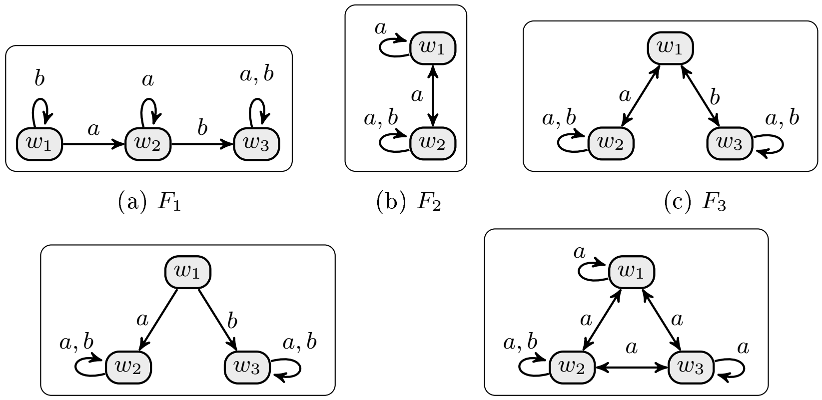
When one studies a notion of knowledge/belief for groups, it is interesting to find out whether it inherits the properties of the knowledge/beliefs of the individuals. For example, suppose that the individual knowledge of all agents in a group is truthful and both positively and negatively introspective. Then, it is well-known that, while the group's common knowledge inherits all these properties, the group's *general* knowledge[7] inherits only truthfulness (i.e., it might not be positively or negatively introspective). Similar studies have been made for notions of belief [1].

This section studies which properties of individual belief are inherited by cautious distributed belief. The discussion is rather semantic, focussing on whether certain frame conditions on individual indistinguishability relations are inherited by the relation that defines cautious distributed belief (see Footnote 4). The connection between these conditions and the properties of knowledge/belief is made thanks to the well-known correspondence between the frame conditions and the validity of certain modal formulas [3, Chapter 3]. Using S for an arbitrary binary relation and $\Box$ ($\Diamond$) for its corresponding normal universal (existential) modality, Table 1 lists some of these frame conditions, together with the formulas that characterise them (and its intuitive epistemic/doxastic reading).[8]

Here are, then, the needed definitions.

[7] A group has general knowledge of φ if and only if everybody in the group knows φ.

[8] More precisely, a *frame* (a model without the valuation) has the given relational property if and only if the formula is valid in the frame (i.e., it is true in any world of the model under any valuation).



(a) F_1 (b) F_2 (c) F_3

(d) F_4 (e) F_5

Figure 1: Counterexamples for the proof of Proposition 3

Definition 4 (Inheriting properties) *Let $x \in \{l, r, t, s, e\}$ be a frame condition, and let $\mathcal{F} \subseteq \{l, r, t, s, e\}$ be a collection of them. Let $G \subseteq A$ be a non-empty set of agents, each one of them associated to a binary relation under a given domain W. A relation $S_G \subseteq W \times W$ defined in terms of the individual relations for agents in G (e.g., their union/intersection)* inherits the condition x under the additional conditions in $\mathcal{F}$ *if and only if S_G has the property x whenever* all *the relations in $\{R_i \mid i \in G\}$ have all the properties in $\mathcal{F} \cup \{x\}$.* ◀

For singleton groups, all properties are preserved. This is because if G is a singleton $\{a\}$, then the cautious distributed belief relation $R^{\forall}_{\{a\}}$ is identical to a's individual relation R_a.

Proposition 3 *Given a collection of relations $\{R_a \subseteq W \times W \mid a \in G\}$ for a group $G \subseteq A$ with at least two agents, the relation $R^{\forall}_G \subseteq W \times W$*

(1) *inherits* seriality *under $\mathcal{F} = \varnothing$;*
(2) *inherits* reflexivity *under $\mathcal{F} = \varnothing$;*
(3) (a) *does not inherit* transitivity *under any $\mathcal{F} \subseteq \{l, e\}$;*
  (b) *inherits* transitivity *under any $\mathcal{F} \supseteq \{r\}$ (also under any $\mathcal{F} \supseteq \{l, s\}$[9]);*
  (c) *inherits* transitivity *under any $\mathcal{F} \supseteq \{s\}$;*
(4) (a) *does not inherit* symmetry *under any $\mathcal{F} \subseteq \{t, e\}$;*
  (b) *does not inherit* symmetry *under any $\mathcal{F} \subseteq \{l, e\}$;*
  (c) *inherits* symmetry *under any $\mathcal{F} \supseteq \{r\}$ (also under any $\mathcal{F} \supseteq \{l, t\}$);*
(5) (a) *does not inherit* Euclidicity *under any $\mathcal{F} \subseteq \{l, s\}$;*
  (b) *does not inherit* Euclidicity *under any $\mathcal{F} \subseteq \{l, t\}$;*

[9] Inheritance under any $\mathcal{F} \subseteq \{l, s\}$ follows immediately from inheritance under any $\mathcal{F} \subseteq \{r\}$, since seriality, transitivity and symmetry together imply reflexivity. The same applies for the properties in (3)(c) and (5)(d) below.

(c) *does not inherit* Euclidicity *under any* $\mathcal{F} \subseteq \{t, s\}$;
(d) *inherits* Euclidicity *under any* $\mathcal{F} \supseteq \{r\}$ *(also under any* $\mathcal{F} \supseteq \{l, t, s\}$*)*.

Proof.

(1) *Pick any* $s \in W$. *Every relation in* $\{R_i \mid i \in G\}$ *is serial so, since* $G \neq \varnothing$, *there is* $a \in G$ *such that* R_a *is serial, and* a *is consistent at* s ($C_a(s) \neq \varnothing$). *Thus,* G *has at least one subgroup* G' *that is maximally consistent at* s *(one containing* a*), and hence there is* $t \in C_{G'}(s) \subseteq C^{\forall}_G(s)$. *Then,* $R^{\forall}_G$ *is serial.*
(2) *Pick any* $s \in W$. *Every relation in* $\{R_i \mid i \in G\}$ *is reflexive, so* $s \in C_a(s)$ *for every* $a \in G$, *and then the only maximally consistent subgroup is* G *itself. Thus,* $C_G(s) = C^{\forall}_G(s)$ *and therefore* $s \in C^{\forall}_G(s)$. *Then,* $R^{\forall}_G$ *is reflexive.*
(3) *(a)* *In frame* F_1 *(Figure 1a), relations* R_a *and* R_b *are transitive, serial and Euclidean. Still,* $R^{\forall}_{\{a,b\}} = \{(w_1, w_1), (w_1, w_2), (w_2, w_2), (w_2, w_3), (w_3, w_3)\}$ *is not transitive.*
(b) *Pick any* $s, t, u \in W$ *such that* $sR^{\forall}_G t$ *and* $tR^{\forall}_G u$. *By reflexivity,* G *is the only maximally consistent subgroup at both* s *and* t, *so* $C_G(s) = C^{\forall}_G(s)$ *and* $C_G(t) = C^{\forall}_G(t)$. *Then,* sR_it *and* tR_iu *for every* $i \in G$, *which by transitivity implies* sR_iu *for all such* i. *Thus,* $u \in C^{\forall}_G(s)$ *and hence* $sR^{\forall}_G u$. *Then,* $R^{\forall}_G$ *is transitive.*
(c) *Pick any* $s, t, u \in W$ *such that* $sR^{\forall}_G t$ *and* $tR^{\forall}_G u$. *Then, there are* $H_1 \subseteq^{max}_s G$ *and* $H_2 \subseteq^{max}_t G$ *such that* $t \in C_{H_1}(s)$ *and* $u \in C_{H_2}(t)$. *By individual symmetry,* $s \in C_{H_1}(t)$ *and* $t \in C_{H_2}(u)$; *then, by individual transitivity,* $t \in C_{H_1}(t)$ *and* $t \in C_{H_2}(t)$. *But then,* $H_1 \cup H_2$ *is consistent at* t *and, since* H_2 *is maximally consistent at* t, *then* $(H_1 \cup H_2) \subseteq H_2$, *that is,* $H_1 \subseteq H_2$. *Hence, the previous* $u \in C_{H_2}(t)$ *implies* $u \in C_{H_1}(t)$ *which, together with* $t \in C_{H_1}(s)$ *and individual transitivity implies* $u \in C_{H_1}(s)$. *Finally, since* H_1 *is maximally consistent at* s *w.r.t.* G, $u \in C^{\forall}_G(s)$, *and hence* $sR^{\forall}_G u$.
(4) *(a)* *In frame* F_2 *(Figure 1b), relations* R_a *and* R_b *are symmetric, transitive and Euclidean. Still,* $R^{\forall}_{\{a,b\}} = \{(w_1, w_1), (w_1, w_2), (w_2, w_2)\}$ *is not symmetric.*
(b) *In frame* F_3 *(Figure 1c), relations* R_a *and* R_b *are symmetric, serial and Euclidean. Still,* $R^{\forall}_{\{a,b\}} = \{(w_1, w_2), (w_1, w_3), (w_2, w_2), (w_3, w_3)\}$ *is not symmetric.*
(c) *Pick any* $s, t \in W$ *such that* $sR^{\forall}_G t$. *By reflexivity,* G *is the only maximally consistent subgroup at both* s *and* t, *so* $C_G(s) = C^{\forall}_G(s)$ *and* $C_G(t) = C^{\forall}_G(t)$. *Then,* sR_it *for every* $i \in G$, *which by symmetry implies* tR_is *for all such* i. *Thus,* $s \in C^{\forall}_G(t)$ *and hence* $tR^{\forall}_G s$. *Then,* $R^{\forall}_G$ *is symmetric.*
(5) *(a)* *In frame* F_3 *(Figure 1c), relations* R_a *and* R_b *are Euclidean, serial and symmetric. Still,* $R^{\forall}_{\{a,b\}} = \{(w_1, w_2), (w_1, w_3), (w_2, w_2), (w_3, w_3)\}$ *is not Euclidean.*
(b) *In frame* F_4 *(Figure 1d), relations* R_a *and* R_b *are Euclidean, serial and transitive. Still,* $R^{\forall}_{\{a,b\}} = \{(w_1, w_2), (w_1, w_3), (w_2, w_2), (w_3, w_3)\}$ *is not Euclidean.*
(c) *In frame* F_5 *(Figure 1e), relations* R_a *and* R_b *are Euclidean, symmetric and transitive. Still,* $R^{\forall}_{\{a,b\}} = (W \times W) \setminus \{(w_2, w_1), (w_2, w_3)\}$ *is not Euclidean.*

(d) *Pick any $s, t, u \in W$ such that $sR_G^\forall t$ and $sR_G^\forall u$. By reflexivity, G is the only maximally consistent subgroup at both s and t, so $C_G(s) = C_G^\forall(s)$ and $C_G(t) = C_G^\forall(t)$. Then, sR_it and sR_iu for every $i \in G$, which by Euclidicity implies tR_iu for all such i. Thus, $u \in C_G^\forall(t)$ and hence $tR_G^\forall u$. Then, $R_G^\forall$ is Euclidean.* ■

Thus, seriality and reflexivity are each inherited without additional assumptions. Symmetry and Euclidicity are both inherited in the presence of reflexivity; transitivity is inherited in the presence of reflexivity, but also in the presence of symmetry. Thus, just as with *individual* belief, cautious distributed belief is factive in reflexive models, and it is consistent in serial models. However, it does not need to be introspective (neither positively nor negatively), even when the model has the frame condition (transitivity and Euclidicity, respectively).

These results are quite different from the corresponding ones for the standard notion of distributed belief. In fact, with the exception of reflexive models (in which cautious and standard distributed belief coincide; see Proposition 4 below), the behaviour of cautious distributed belief is, in this respect, the opposite of that of standard distributed belief. For the latter, transitivity, symmetry and Euclidicity are each inherited without additional assumptions, while seriality is is inherited only in the presence of reflexivity [1].

4 Relationship between D_G and $D_G^\forall$

This section discusses the relationship between standard and cautious distributed belief. The following definitions will be useful.

Definition 5 *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two languages whose formulas can be evaluated over pointed belief models.*

- *$\mathcal{L}_2$ is* at least as expressive *as $\mathcal{L}_1$ (notation: $\mathcal{L}_1 \preccurlyeq \mathcal{L}_2$) if and only if every formula in $\mathcal{L}_1$ has a semantically equivalent formula in $\mathcal{L}_2$: for every $\alpha_1 \in \mathcal{L}_1$ there is $\alpha_2 \in \mathcal{L}_2$ s.t., for every pointed belief model $(\mathcal{M}, s)$, we have $\mathcal{M}, s \vDash \alpha_1$ if and only if $\mathcal{M}, s \vDash \alpha_2$.*[10]
- *$\mathcal{L}_1$ and $\mathcal{L}_1$ are* equally expressive *(notation: $\mathcal{L}_1 \approx \mathcal{L}_2$) if and only if $\mathcal{L}_1 \preccurlyeq \mathcal{L}_2$ and $\mathcal{L}_2 \preccurlyeq \mathcal{L}_1$.*
- *$\mathcal{L}_2$ is* strictly more expressive *than $\mathcal{L}_1$ (notation: $\mathcal{L}_1 \prec \mathcal{L}_2$) if and only if $\mathcal{L}_1 \preccurlyeq \mathcal{L}_2$ and $\mathcal{L}_2 \not\preccurlyeq \mathcal{L}_1$.*[11] ◀

The proposition below provides some connections between D_G and $D_G^\forall$. First, $D_G^\forall$ is definable in terms of D_G and Boolean operators. Second, both notions coincide when the indistinguishability relations are reflexive.

[10] A typical strategy for proving $\mathcal{L}_1 \preccurlyeq \mathcal{L}_2$ is to give a translation $tr : \mathcal{L}_1 \to \mathcal{L}_2$ such that for every $(\mathcal{M}, s)$ we have $\mathcal{M}, s \vDash \alpha_1$ iff $\mathcal{M}, s \vDash tr(\alpha_1)$. The crucial cases are those for the operators in $\mathcal{L}_1$ that do not occur in $\mathcal{L}_2$.

[11] A typical strategy for proving $\mathcal{L}_1 \not\preccurlyeq \mathcal{L}_2$ is to find two pointed models that satisfy exactly the same formulas in $\mathcal{L}_2$, and yet can be distinguished by a formula in $\mathcal{L}_1$.

Proposition 4

(1) $\vDash D^{\forall}_G\varphi \leftrightarrow \bigwedge_{G'\subseteq G}\Big((\neg D_{G'}\bot \wedge \bigwedge_{G'\subset H\subseteq G} D_H\bot) \to D_{G'}\varphi\Big)$.

(2) *Let* $\mathbf{T}$ *be the class of all belief models whose accessibility relations are all reflexive. Then,* $\mathbf{T} \vDash D^{\forall}_G\varphi \leftrightarrow D_G\varphi$.

Proof.

(1) *Suppose* $\mathcal{M}, s \vDash D^{\forall}_G\varphi$. *By definition, this is the case if and only if every* $G' \subseteq^{max}_s G$ *is such that* $\mathcal{M}, s \vDash D_{G'}\varphi$. *But the fact that* $G' \subseteq^{max}_s G$ *(i.e.,* G' *is a maximally consistent subgroup of* G *at* s*) is equivalently stated as* $\mathcal{M}, s \vDash \neg D_{G'}\bot \wedge \bigwedge_{G'\subset H\subseteq G} D_H\bot$.[12] *Then, the previous is the case if and only if* $\mathcal{M}, s \vDash \bigwedge_{G'\subseteq G}\big((\neg D_{G'}\bot \wedge \bigwedge_{G'\subset H\subseteq G} D_H\bot) \to D_{G'}\varphi\big)$.

(2) *Immediate, as* $C_G(s) = C^{\forall}_G(s)$ *holds for any reflexive belief model* $\mathcal{M}$*, world* $s \in \mathrm{D}(\mathcal{M})$ *and group* $\varnothing \neq G \subseteq A$ *(see the proof of Proposition 3(2)).* ■

Using the first part of Proposition 4, one can define a translation that takes any formula in $\mathcal{L}_{D^{\forall}}$ and returns a semantically equivalent formula in $\mathcal{L}_D$. Thus, it already establishes a connection between $\mathcal{L}_D$ and $\mathcal{L}_{D^{\forall}}$.

Corollary 1 *$\mathcal{L}_D$ is at least as expressive as $\mathcal{L}_{D^{\forall}}$ (in symbols: $\mathcal{L}_{D^{\forall}} \preccurlyeq \mathcal{L}_D$).* ■

A question remains: is $\mathcal{L}_{D^{\forall}}$ also at least as expressive as $\mathcal{L}_D$ (so the languages are equally expressive), or is $\mathcal{L}_D$ *strictly* more expressive than $\mathcal{L}_{D^{\forall}}$ (so there are situations that $\mathcal{L}_{D^{\forall}}$ cannot tell apart, and yet they can be distinguished by $\mathcal{L}_D$)?

When discussing the relative expressivity of modal languages, it is useful to have a semantic notion guaranteeing that two pointed models cannot be distinguished by a language. A multi-agent version of the standard notion of bisimulation (see, e.g., [3, Section 2.2]) plays this role for the basic multi-agent epistemic language. When the modality for standard distributed knowledge is added (i.e., for $\mathcal{L}_D$), one rather requires the notion of *collective bisimulation* [17], which asks for the conditions of the standard bisimulation to be fulfilled by the *intersection relation* of every group. Still, the results below will show that this notion is not the adequate one for our language $\mathcal{L}_{D^{\forall}}$.

The notion of $\mathcal{L}_{D^{\forall}}$-bisimulation defined below will be shown to be the adequate one for $\mathcal{L}_{D^{\forall}}$: it implies that two pointed models cannot be distinguished by $\mathcal{L}_{D^{\forall}}$ (Proposition 5), and it exists between any image-finite pointed models that cannot be distinguished by the language (Proposition 6).

Definition 6 ($\mathcal{L}_{D^{\forall}}$-Bisimulation) *Let* $\mathcal{M} = \langle W, R, v\rangle$ *and* $\mathcal{M}' = \langle W', R', v'\rangle$ *be two belief models. A non-empty relation* $Z \subseteq \mathrm{D}(\mathcal{M}) \times \mathrm{D}(\mathcal{M}')$ *is a* $\mathcal{L}_{D^{\forall}}$-bisimulation *between* $\mathcal{M}$ *and* $\mathcal{M}'$ *if and only if* Zss' *implies all of the following.*

Atom. *For all* $p \in P$*:* $s \in v(p)$ *if and only if* $s' \in v'(p)$*.*

Forth. *For all* $G \subseteq A$*, for all* $t \in \mathrm{D}(\mathcal{M})$*: if there is* $H \subseteq^{max}_s G$ *such that* $t \in C_H(s)$*, then there are* $H' \subseteq^{max}_{s'} G$ *and* $t' \in C_{H'}(s')$ *such that* Ztt'*.*[13]

[12] Note: this relies on the fact that G is finite (because A is finite).

[13] Equivalently: for all $G \subseteq A$, for all $t \in \mathrm{D}(\mathcal{M})$, if $sR^{\forall}_G t$, then $\exists t'$ such that $s'R^{\forall}_G t'$ and Ztt'.

Back. *For all $G \subseteq A$, for all $t' \in \mathrm{D}(\mathcal{M}')$: if there is $H' \subseteq^{max}_{s'} G$ such that $t' \in C_{H'}(s')$, then there are $H \subseteq^{max}_{s} G$ and $t \in C_H(s)$ such that Ztt'.*[14]

Write $Z : \mathcal{M}, s \leftrightarrows_{D^\forall} \mathcal{M}', s'$ when Z is a $\mathcal{L}_{D^\forall}$-bisimulation between $\mathcal{M}$ and $\mathcal{M}'$ with Zss'. Write $\mathcal{M}, s \leftrightarrows_{D^\forall} \mathcal{M}', s$ when there is such a bisimulation Z. ◀

A $\mathcal{L}_{D^\forall}$-bisimulation follows the idea of a standard one. First, $\mathcal{L}_{D^\forall}$-bisimilar worlds should satisfy the same atoms. Then, if one of them has a 'relevant successor' t, the other should also have a 'relevant successor' t' and, moreover, these successors should be $\mathcal{L}_{D^\forall}$-bisimilar. The only difference between a $\mathcal{L}_{D^\forall}$-bisimulation and others in the literature is what 'a relevant successor' means. In a multi-agent standard bisimulation, a 'relevant successor' is any world that can be reached through the relation R_i, for some agent $i \in A$. In a collective bisimulation, a 'relevant successor' is any world that can be reached through the *intersection of the relations* of the individuals in G, for some group $G \subseteq A$. In the just defined $\mathcal{L}_{D^\forall}$ bisimulation, a 'relevant successor' is any world that belongs to the conjecture set of some maximally consistent subgroup of G, for some non-empty set of agents $G \subseteq A$.[15] As it is shown below, this definition guarantees that every world in W that is relevant for cautious distributed belief in $(\mathcal{M}, s)$ has a 'matching' world in W' that is relevant for cautious distributed belief in $(\mathcal{M}', s')$ (and vice versa). (For an example of $\mathcal{L}_{D^\forall}$-bisimilar models see the proof of fact 2 below.)

Definition 7 ($\mathcal{L}_{D^\forall}$-equivalence) *Two pointed models $\mathcal{M}, s$ and $\mathcal{M}', s'$ are $\mathcal{L}_{D^\forall}$-equivalent (notation: $\mathcal{M}, s \leftsquigarrow\!\!\rightsquigarrow_{D^\forall} \mathcal{M}', s'$) if and only if, for every $\varphi \in \mathcal{L}_{D^\forall}$,*

$$\mathcal{M}, s \vDash \varphi \quad \text{if and only if} \quad \mathcal{M}', s' \vDash \varphi.$$

When the models are clear from context, we will write simply $s \leftsquigarrow\!\!\rightsquigarrow_{D^\forall} s'$. ◀

Proposition 5 ($\mathcal{L}_{D^\forall}$-Bisimilarity implies $\mathcal{L}_{D^\forall}$-equivalence) *Let $\mathcal{M}, s$ and $\mathcal{M}', s'$ be pointed belief models. Then,*

$$\mathcal{M}, s \leftrightarrows_{D^\forall} \mathcal{M}', s' \quad \text{implies} \quad \mathcal{M}, s \leftsquigarrow\!\!\rightsquigarrow_{D^\forall} \mathcal{M}', s'. \qquad \blacksquare$$

Proof. First, pull out the universal quantification over formulas hidden in $\leftsquigarrow\!\!\rightsquigarrow_{D^\forall}$, so the statement becomes "for every formula in $\mathcal{L}_{D^\forall}$: if two pointed models are $D^\forall$-bisimilar, then they agree on the formula's truth-value"*. Now, proceed by structural induction on formulas in $\mathcal{L}_{D^\forall}$. The case for atomic propositions follows from the* **atom** *clause, and those for Boolean operators (in our case, $\neg$ and $\wedge$) follow from their respective inductive hypotheses.*

For formulas expressing cautious distributed belief, work by contraposition. ($\Rightarrow$) Suppose $\mathcal{M}', s' \nvDash D^\forall_G \varphi$. Then, there are $H' \subseteq^{max}_{s'} G$ and $t' \in C_{H'}(s')$

[14] Equivalently: for all $G \subseteq A$, for all $t' \in \mathrm{D}(\mathcal{M}')$, if $s' R^\forall_G t'$, then $\exists t$ such that $s R^\forall_G t$ and Ztt'.

[15] Note then that, while a collective bisimulation requires that a group is inconsistent at any world bisimilar to one at which the group is inconsistent, this not the case for a $\mathcal{L}_{D^\forall}$-bisimulation. The models in the proof of Fact 2 below show this.

such that $\mathcal{M}', t' \nvDash \varphi$. But $\mathcal{M}, s \leftrightarroweq_{D^\forall} \mathcal{M}', s'$ so, by the **back** *clause, there are $H \subseteq_s^{max} G$ and $t \in C_H(s)$ such that $\mathcal{M}, t \leftrightarroweq_{D^\forall} \mathcal{M}', t'$. By inductive hypothesis, the latter implies $\mathcal{M}, t \leftrightsquigarrow_{D^\forall} \mathcal{M}', t'$, so from the earlier $\mathcal{M}', t' \nvDash \varphi$ it follows that $\mathcal{M}, t \nvDash \varphi$. Thus, $\mathcal{M}, s \nvDash D_G^\forall \varphi$. ($\Leftarrow$) Similar, using the* **forth** *clause instead.* ■

A weakened version of the converse holds: if two *image-finite* pointed belief models are $\mathcal{L}_{D^\forall}$-equivalent, then there is a $\mathcal{L}_{D^\forall}$-bisimulation between them.

Proposition 6 ($\mathcal{L}_{D^\forall}$-Equivalence implies $\mathcal{L}_{D^\forall}$-bisimilarity) *Let $\mathcal{M}, s$ and $\mathcal{M}', s'$ be* image-finite *pointed belief models.*[16] *Then,*

$$\mathcal{M}, s \leftrightsquigarrow_{D^\forall} \mathcal{M}', s' \quad \textit{implies} \quad \mathcal{M}, s \leftrightarroweq_{D^\forall} \mathcal{M}', s'.$$

Proof. *It will be shown that $\leftrightsquigarrow_{D^\forall}$ is in fact a $\mathcal{L}_{D^\forall}$-bisimulation. To do this, take any s and s' such that $s \leftrightsquigarrow_{D^\forall} s'$; it will be shown that the three clauses of Definition 6 are satisfied.*

- ***Atom.*** *It is clear that s and s' satisfy the same atomic propositions.*
- ***Forth.*** *Take any $\varnothing \subset G \subseteq A$; suppose there are $H \subseteq_s^{max} G$ and $t \in C_H(s)$. For the sake of a contradiction, suppose there are no $H' \subseteq_{s'}^{max} G$ and $t' \in C_{H'}(s')$ such that $t \leftrightsquigarrow_{D^\forall} t'$; in other words, suppose that every $H' \subseteq_{s'}^{max} G$ and $t' \in C_{H'}(s')$ are such that $t \not\leftrightsquigarrow_{D^\forall} t'$. This means that if $t'_i \in C_G^\forall(s')$ then $t \not\leftrightsquigarrow_{D^\forall} t'_i$: for every world $t'_i \in C_G^\forall(s')$ there is $\psi_i \in \mathcal{L}_{D^\forall}$ such that $\mathcal{M}, t \nvDash \psi_i$ and $\mathcal{M}', t'_i \vDash \psi_i$.*
  Now note that $C_G^\forall(s')$ is non-empty and finite.[17] *Thus, $\psi := \bigvee_{t'_i \in C_G^\forall(s')} \psi_i$ is a non-contradictory formula (as $C_G^\forall(s')$ is non-empty) in $\mathcal{L}_{D^\forall}$ (as $C_G^\forall(s')$ is finite). Hence, $\mathcal{M}, t \nvDash \psi$ and yet $\mathcal{M}', t'_i \vDash \psi$ for every $t'_i \in C_G^\forall(s')$. Since $H \subseteq_s^{max} G$ and $t \in C_H(s)$, the former implies $\mathcal{M}, s \nvDash D_G^\forall \psi$; nevertheless, the latter implies $\mathcal{M}', s' \vDash D_G^\forall \psi$. This contradicts the original assumption $s \not\leftrightsquigarrow_{D^\forall} s'$. Therefore, there is some $H' \subseteq_{s'}^{max} G$ and some $t' \in C_{H'}(s')$ such that $t \leftrightsquigarrow_{D^\forall} t'$.*
- ***Back.*** *Analogous to the previous clause.* ■

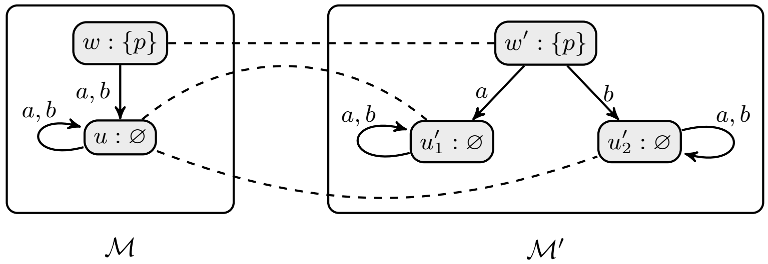
We have now enough tools to answer the question above.

Fact 2 *$\mathcal{L}_{D^\forall}$ is* not *at least as expressive as $\mathcal{L}_D$ (in symbols: $\mathcal{L}_D \not\preccurlyeq \mathcal{L}_{D^\forall}$).*

Proof. *Consider the belief models shown below.*

[16] A belief model $\mathcal{M}$ is image-finite iff $C_a(s)$ is finite for every $s \in \mathrm{D}(\mathcal{M})$ and every $a \in A$ (equivalently, iff $C_G(s)$ is finite for every $s \in \mathrm{D}(\mathcal{M})$ and every $G \subseteq A$).

[17] It is non-empty because, from $H \subseteq_s^{max} G$ and $t \in C_H(s)$, it follows that $t \in C_a(s)$ for some $a \in H \subseteq G$, and thus $\mathcal{M}, s \vDash \neg D_a^\forall \bot$. But $s \leftrightsquigarrow_{D^\forall} s'$, so $\mathcal{M}', s' \vDash \neg D_a^\forall \bot$, so a is consistent at s' in $\mathcal{M}'$. Then, since a is in G, there should be an $H' \subseteq_{s'}^{max} G$ with $a \in H'$. But, once again, a is consistent, so $C_{H'}(s') \neq \varnothing$ and thus $C_G^\forall(s') \neq \varnothing$. It is finite because the models are image-finite.



■

Use $MC_G(s)$ to denote all subgroups of G that are maximally consistent at s. The dashed edges define a bisimulation between $\mathcal{M}, w$ and $\mathcal{M}', w'$. Indeed,

- $\boldsymbol{(w, w')}$. *The* ***atom*** *clause is immediate. Now* ***forth***. *For $G = \{a\}$, note that $MC_{\{a\}}(w) = \{\{a\}\}$ and thus $C^{\forall}_{\{a\}}(w) = \{u\}$. But then $MC_{\{a\}}(w') = \{\{a\}\}$ and thus $C^{\forall}_{\{a\}}(w') = \{u_1'\}$; moreover, Zuu_1'. The case for $G = \{b\}$ is analogous. For $G = \{a, b\}$, note that $MC_{\{a,b\}}(w) = \{\{a, b\}\}$ and thus $C^{\forall}_{\{a,b\}}(w) = \{u\}$. But then $MC_{\{a,b\}}(w') = \{\{a\}, \{b\}\}$ and thus $C^{\forall}_{\{a,b\}}(w') = \{u_1', u_2'\}$; moreover, Zuu_1' and Zuu_2'. The* ***back*** *clause follows a similar pattern.*
- $\boldsymbol{(u, u_1')}$. *The* ***atom*** *clause is immediate. Consider* ***forth***. *For $G = \{a\}$, note that $MC_{\{a\}}(u) = \{\{a\}\}$ and thus $C^{\forall}_{\{a\}}(u) = \{u\}$. But then $MC_{\{a\}}(u_1') = \{\{a\}\}$ and thus $C^{\forall}_{\{a\}}(u') = \{u_1'\}$; moreover, Zuu_1'. The case for $G = \{b\}$ is analogous. For $G = \{a, b\}$, note that $MC_{\{a,b\}}(u) = \{\{a, b\}\}$ and thus $C^{\forall}_{\{a,b\}}(u) = \{u\}$. But then $MC_{\{a,b\}}(u_1') = \{\{a, b\}\}$ and thus $C^{\forall}_{\{a,b\}}(u_1') = \{u_1'\}$; moreover, Zuu_1'. The* ***back*** *clause follows a similar pattern.*
- $\boldsymbol{(u, u_2')}$. *As the previous case.*

Thus, $M, w \leftrightarrows_{D^{\forall}} \mathcal{M}', w'$ and hence, by Proposition 5, $\mathcal{M}, w \leftrightsquigarrow_{D^{\forall}} \mathcal{M}', w'$. However, the pointed models can be distinguished by a formula in $\mathcal{L}_D$, as $\mathcal{M}, w \not\models D_{\{a,b\}} \bot$ and yet $\mathcal{M}', w' \models D_{\{a,b\}} \bot$. Therefore $\mathcal{L}_D \not\preccurlyeq \mathcal{L}_{D^{\forall}}$. ■

Note how the belief models used above are serial, transitive and Euclidean: the kind of models one normally uses for representing a proper notion of belief.

Corollary 2 *$\mathcal{L}_D$ is strictly more expressive than $\mathcal{L}_{D^{\forall}}$ (symbols: $\mathcal{L}_{D^{\forall}} \prec \mathcal{L}_D$).* ■

Thus, $\mathcal{L}_{D^{\forall}}$ can 'see' strictly less than what $\mathcal{L}_D$ can. The proposition below shows that the group inconsistency constant $\asymp_G$ introduced before is exactly what the former needs to 'see' exactly as much as the latter.

Proposition 7 *$\mathcal{L}_{D^{\forall},\asymp}$ and $\mathcal{L}_D$ are equally expressive (symbols: $\mathcal{L}_{D^{\forall},\asymp} \approx \mathcal{L}_D$).*

Proof. *Clearly, $\models \asymp_G \leftrightarrow D_G \bot$. Thus, both $\asymp_G$ and $D^{\forall}_G$ are definable in $\mathcal{L}_D$ (for the latter, recall Proposition 4), so $\mathcal{L}_{D^{\forall},\asymp} \preccurlyeq \mathcal{L}_D$.*

For proving $\mathcal{L}_D \preccurlyeq \mathcal{L}_{D^{\forall},\asymp}$, it is enough to show that D_G is definable in $\mathcal{L}_{D^{\forall},\asymp}$:

$$\models D_G\, \varphi \leftrightarrow (\asymp_G \vee D^{\forall}_G\, \varphi).$$

($\Rightarrow$) Suppose $\mathcal{M}, s \models D_G\,\varphi$, so every $t \in C_G(s)$ is such that $\mathcal{M}, t \models \varphi$. Assume further that $\mathcal{M}, s \not\models \asymp_G$. Then, $MC_G(s) = \{G\}$ and thus $\mathcal{M}, s \models D^{\forall}_G\,\varphi$. ($\Leftarrow$) Proceed by contraposition: suppose $\mathcal{M}, s \not\models D_G\,\varphi$. Then, there is $t \in C_G(s)$ such that $\mathcal{M}, t \not\models \varphi$. Thus, $C_G(s) \neq \varnothing$, so $MC_G(s) = \{G\}$. From the former, $\mathcal{M}, s \not\models \asymp_G$; from $t \in C_G(s)$ and the latter, $\mathcal{M}, s \not\models D^{\forall}_G\,\varphi$. Thus, $\mathcal{M}, s \not\models \asymp_G \vee D^{\forall}_G\,\varphi$. ■

5 Summary and further work

This paper has introduced the notion of *cautious* distributed belief. While a set of agents G has distributed belief that φ ($D_G\,\varphi$) if and only if φ is true in every world in the conjecture set of the group, the group has cautious distributed belief that φ ($D^{\forall}_G\,\varphi$) if and only if φ is true in every world in the conjecture set of *every* maximally consistent subgroup of G.

The paper has discussed basic properties of $D^{\forall}$, showing, e.g., how it is inconsistent if and only if all agents in the group are inconsistent. Then, the paper has studied whether this group notion inherits properties from the individual notions of the group's members. It has been shown that consistency and truthfulness (technically, seriality and reflexivity) are inherited, and that so are both positive and negative introspection (technically, transitivity and Euclidicity) when the epistemic/doxastic notion is also truthful (technically, reflexive). This is the opposite of what happens with standard distributed belief, which inherits both positive and negative introspection (transitivity, symmetry and Euclidicity) without additional assumptions, and inherits consistency (seriality) only when the individual notions are truthful (reflexive). The final part of the paper has focussed on the relationship between $D^{\forall}_G$ and D_G. It has been show that, while they coincide in reflexive models (i.e., cautions distributed *knowledge* coincides with standard distributed *knowledge*), in general the latter (D) is strictly more expressive than the former ($D^{\forall}$). This difference in expressivity has been proved by providing a notion of structural equivalence that, within image-finite models, characterises modal equivalence w.r.t to $\mathcal{L}_{D^{\forall}}$ (a language extending the propositional one with $D^{\forall}$). Finally, the paper has identified the 'missing piece' that makes a language with $D^{\forall}$ as expressive as one with D.

Among the questions that still need answer, the main ones are an axiom system for the language $\mathcal{L}_{D^{\forall}}$ and a study of its complexity profile. Among the further research lines, the idea of dealing with potential group inconsistencies by looking at maximally consistent subgroups leads to another interesting alternative: a group has *bold* distributed belief that φ (say, $D^{\exists}_G\,\varphi$) if and only if φ is true in every world in the conjecture set of *some* maximally consistent subgroup of G. The quantification pattern of this alternative notion ($\exists\forall$) suggest that, different from $D^{\forall}$, the bold distributed belief operator is not a normal modal operator. Thus, further technical tools will be needed for studying its profile.

References

1. Ågotnes, T., Wáng, Y.N.: Group belief. Journal of Logic and Computation **31**(8), 1959–1978 (2021). https://doi.org/10.1093/logcom/exaa068
2. van Benthem, J., Pacuit, E.: Dynamic logics of evidence-based beliefs. Studia Logica **99**(1), 61–92 (2011). https://doi.org/10.1007/s11225-011-9347-x
3. Blackburn, P., de Rijke, M., Venema, Y.: Modal Logic. Cambridge University Press (2002)
4. de Bruin, B.: Explaining Games: The Epistemic Programme in Game Theory, Synthese Library Series, vol. 346. Springer, Dordrecht (2010)
5. Dung, P.M.: On the acceptability of arguments and its fundamental role in nonmonotonic reasoning, logic programming and n-person games. Artificial Intelligence **77**(2), 321–358 (1995). https://doi.org/10.1016/0004-3702(94)00041-X
6. Fagin, R., Halpern, J.Y., Moses, Y., Vardi, M.Y.: Reasoning about knowledge. The MIT Press, Cambridge, Mass. (1995)
7. Halpern, J.Y., Moses, Y.: A guide to the modal logics of knowledge and belief: Preliminary draft. In: Joshi, A.K. (ed.) Proceedings of the 9th International Joint Conference on Artificial Intelligence. Los Angeles, CA, USA, August 1985. pp. 480–490. Morgan Kaufmann (1985), http://ijcai.org/Proceedings/85-1/Papers/094.pdf
8. Halpern, J.Y., Moses, Y.: Knowledge and common knowledge in a distributed environment. Journal of the ACM **37**(3), 549–587 (1990). https://doi.org/10.1145/79147.79161
9. Hendricks, V.F.: Introduction: 8 bridges between mainstream and formal epistemology. Philosophical Studies **128**(1), 1–5 (2006). https://doi.org/10.1007/s11098-005-4068-5
10. Herzig, A., Lorini, E., Perrotin, E., Romero, F., Schwarzentruber, F.: A logic of explicit and implicit distributed belief. In: ECAI 2020 - 24th European Conference on Artificial Intelligence. Frontiers in Artificial Intelligence and Applications, vol. 325, pp. 753–760. IOS Press (2020). https://doi.org/10.3233/FAIA200163
11. Hilpinen, R.: Remarks on personal and impersonal knowledge. Canadian Journal of Philosophy **7**(1), 1–9 (1977). https://doi.org/10.1080/00455091.1977.10716173
12. Hintikka, J.: Knowledge and Belief: An Introduction to the Logic of the Two Notions. Cornell University Press, Ithaca, N.Y. (1962)
13. Lewis, D.: Convention: A Philosophical Study. Harvard University Press, Cambridge, Massachusetts (1969)
14. Meyer, J.J.C., van der Hoek, W.: Epistemic Logic for AI and Computer Science. Cambridge University Press, New York, N.Y., U.S.A. (1995). https://doi.org/10.1017/CBO9780511569852
15. Perea, A.: Epistemic Game Theory: Reasoning and Choice. Cambridge University Press, Cambridge (2012)
16. Reiter, R.: A logic for default reasoning. Artificial Intelligence **13**(1-2), 81–132 (1980). https://doi.org/10.1016/0004-3702(80)90014-4
17. Roelofsen, F.: Bisimulation and distributed knowledge revisited. Journal of Applied Non-Classical Logics **17**, 255–273 (01 2007). https://doi.org/10.3166/jancl.17.255-273

A Stit Logic of Intentionality

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Abstract. We extend epistemic stit theory with a modality $I_\alpha\varphi$, meant to express that at some moment agent α had a present-directed intention toward the realization of φ. The semantics is based on the extension of stit frames with special topologies associated to agents. The open sets of the associated topology are interpreted as present-directed intentions, that support whether an agent had an intention of realizing a specific state of affairs when it chose one of its available actions and executed it. As an important application, we use $I_\alpha\varphi$ to formalize intentional action and intentional responsibility. We present an axiom system for our logic of intentionality, and prove that it is sound and complete.

Keywords: Stit Logic · Logic of Action · Logic of Intention

1 Introduction

Suppose that you are a lawyer. You are part of the prosecution in a trial where the defendant is being accused of murder. The case is as follows: while driving, the defendant ran over and killed a traffic officer who was standing at a crossing walk. At the trial, the defense is seeking for a charge of only *involuntary manslaughter*, while you and the prosecution contend that it was either *second-* or *first-degree murder*. This means that the verdict revolves around the intentionality of the defendant. If the prosecuting team—to which you belong—is able to provide sufficient evidence for claiming that the defendant had an intention to kill the traffic officer, then the verdict would be either of second- or first-degree murder—according to whether the murder was either planned or unplanned. If the defense shows that the evidence does not support that there was an intention to kill—as would be the case if, for instance, the defendant was drunk while driving and had no real motive for killing the traffic officer—then the verdict would be of only manslaughter.

This example shows that, at least in criminal law, intentionality is of the utmost importance. For many reasons, this importance has carried over to philosophy, giving rise to a complex field of research. In the opening lines of *Stanford Encyclopedia of Philosophy*'s current entry for intention, [28] writes:

> Philosophical perplexity about intention begins with its appearance in three guises: intention for the future, as when we intend to complete

this entry by the end of the month; the intention with which someone acts, as I am typing with the further intention of writing an introductory sentence; and intentional action, as in the fact that I am typing these words intentionally.

Across the philosophical literature, it is well-known that modelling intentionality is difficult, that it leads to many interesting discussions, and that no camp has the last word on what the best framework for analyzing the concept is. However, most authors agree with the quote above, and thus identify three main forms of intentionality:

1. *Future-directed intentions*: following the interpretation of [7] and [8], future-directed intentions are elements in plans that agents make. In the quote above, when the author mentions that he intends to complete his entry by the end of the month, the word *intends* refers to future-directed intentions. The literature also acknowledges the existence of so-called *present-directed intentions*, referring to mental states that regard what agents intend to do now.
2. *Intentional action*: following [10], who offered a precise account of claims advanced by [3], intentional action is a mode of acting. In the quote above, when the author mentions that he types words while writing his entry, and that he is doing so intentionally, he is referring to the intentional action of hitting the keys in the keyboard.
3. *Intention-with-which*: following [14], intention-with-which is a description of the primary reason that an agent has for acting in a specific way. In the quote above, when the author mentions that he types words toward the goal of writing an introductory sentence, then writing an introductory sentence is the intention-with-which he types.

The main problem in philosophy of intention, then, has been to find unity in these three senses of intentionality. According to [28], such an endeavor matters for questions in philosophy of mind, but also for ethics, for epistemology, and for the nature of practical reason. Intuitively speaking, neither of these three "guises" of intentionality is the same as another, but they are all closely related.

To address the challenge of incorporating intentions—seen either as mental states or as modes of acting—into the stit-theoretic conception of agency, we use *present-directed intentions*. The idea is to associate a special topology to each agent. In any such associated topology, all the open sets are dense,[3] and they represent the agent's present-directed intentions—written "p-d intentions," from here on—at the moment of acting. Thus, if an open set U in the topology associated to α supports φ ($U \subseteq \varphi$), then U is a p-d intention of α toward the realization of φ. Roughly speaking, the proposal for the semantics of the modality $I_\alpha \varphi$, then, is as follows: $I_\alpha \varphi$ holds at an index iff at such an index α

[3] An open set is *dense* in a topology iff it is consistent with all the other non-empty open sets of the topology (see Definition 2).

p-d intended φ. For all practical purposes, then, the conjunction $[\alpha]\varphi \wedge I_\alpha[\alpha]\varphi$ is meant to evoke that α has intentionally seen to it that φ.[4]

Instead of spinning around the concepts informally, let us dive into the formal definitions for our notion of intentionality.

2 A Semantics of Intentionality

We start by reminding the reader of some basic definitions from General Topology. For any other basic definitions that we might be taking for granted, the reader is referred to [30] or [17] as proper background textbooks.

Definition 1 (Topological spaces). *Let X be a set. $\tau \subseteq 2^X$ is called a* topology on X *if it meets the following requirements: (a) $X, \emptyset \in \tau$; (b)*closure under finite intersections*: if $U, V \in \tau$, then $U \cap V \in \tau$; (c)* closure under arbitrary unions*: for a family $\mathcal{G} \subseteq \tau$, $\bigcup \mathcal{G} \in \tau$.*

A topological space*, then, is a pair (X, τ), where X is a set and τ is a topology on X. The elements of τ are called* open sets*. Complements of open sets are called* closed sets*. For $A \subseteq X$, the* interior *of A is defined as the $\subseteq$-largest open set included in A, and will be denoted by $int\ A$. The* closure *of A is defined as the $\subseteq$-least closed set including A, and will be denoted by $Cl(A)$. Standard result in General Topology are that (a) for $x \in X$ and $A \subseteq X$, $x \in int\ A$ iff there exists an open set U such that $x \in U \subseteq A$; and (b) $x \in Cl(A)$ iff every open set U such that $x \in U$ intersects A ($U \cap A \neq \emptyset$).*

Definition 2 (Density). *For a topological space (X, τ) and $A \subseteq X$, A is said to be τ-*dense *in X iff $Cl(A) = X$, or, equivalently, if for every non-empty open set $O \in \tau$, $O \cap A \neq \emptyset$.*

With these basic definitions, let us introduce a logic that we call intentional epistemic stit theory.

Definition 3 (Syntax of intentional epistemic stit theory). *Given a finite set Ags of agent names and a countable set of propositions P, the grammar for the formal language $\mathcal{L}_I$ is given by*

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box\varphi \mid [\alpha]\varphi \mid K_\alpha\varphi \mid I_\alpha\varphi.$$

where $p \in P$ and $\alpha \in Ags$.

$\Box\varphi$ is meant to express the *historical necessity* of φ. $\Diamond\varphi$ abbreviates $\neg\Box\neg\varphi$, and it encodes the historical possibility of φ. $[\alpha]\varphi$ stands for 'agent α has seen to it that φ.' $K_\alpha\varphi$ stands for 'agent α knew φ,' and $I_\alpha\varphi$ expresses that 'α had

[4] Of course, this reading of the modality $I_\alpha\varphi$ and of the conjunction $[\alpha]\varphi \wedge I_\alpha[\alpha]\varphi$ positions our proposal as belonging to a particular philosophical standpoint on the relation between intentions and intentional action—in the context of the discussion, on the trends in philosophy of intention, at the beginning of the present section. We address the details of such a standpoint in Subsection 3.

a p-d intention toward the realization of φ', or that 'α p-d intended φ,' or that 'α p-d intended that φ would hold.'

As for the semantics, the structures on which the formulas of $\mathcal{L}_\mathsf{I}$ are evaluated are based on what we call *intentional epistemic branching-time frames.*

Definition 4 (*Iebt*-frames & models).

A tuple $\left\langle M, \sqsubset, Ags, \mathbf{Choice}, \{\sim_\alpha\}_{\alpha \in Ags}, \tau \right\rangle$ *is called an* intentional epistemic branching-time frame *(*iebt-*frame for short) iff*

- *M is a non-empty set of* moments *and $\sqsubset$ is a strict partial ordering on M satisfying* no backward branching*: for $m, m', m'' \in M$ such that $m' \sqsubset m$ and $m'' \sqsubset m$, either $m' = m''$ or $m' \sqsubset m''$ or $m'' \sqsubset m'$. Each maximal $\sqsubset$-chain is called a* history*. The set of all histories is denoted by H. For $m \in M$, $H_m := \{h \in H; m \in h\}$. Tuples $\langle m, h\rangle$ such that $m \in M$, $h \in H$, and $m \in h$, are called* indices*, and the set of indices is denoted by $I(M \times H)$.*
- **Choice** *is a function that maps each agent α and moment m to a partition $\mathbf{Choice}^m_\alpha$ of H_m, where the cells of such a partition represent α's available choices of action at m. For $m \in M$ and $h \in H_m$, $\mathbf{Choice}^m_\alpha(h)$ denotes the cell of the partition $\mathbf{Choice}^m_\alpha$ that includes h. This cell represents the choice of action that α has performed at index $\langle m, h\rangle$, and I refer to it as α's* current choice of action at $\langle m, h\rangle$. **Choice** *satisfies two conditions:*
  - ($\mathtt{NC}$) No choice between undivided histories*: for $\alpha \in Ags$ and $h, h' \in H_m$, if $m' \in h \cap h'$ for some $m' \sqsupset m$, then $h \in L$ iff $h' \in L$ for every $L \in \mathbf{Choice}^m_\alpha$.*
  - ($\mathtt{IA}$) Independence of agency*: a function s on Ags is called a* selection function *at m if it assigns to each α a member of $\mathbf{Choice}^m_\alpha$. If $\mathbf{Select}^m$ denotes the set of all selection functions at m, then, for $m \in M$ and $s \in \mathbf{Select}^m$, $\bigcap_{\alpha \in Ags} s(\alpha) \neq \emptyset$. This condition establishes that concurrent actions by distinct agents must be independent: the choices of action of a given agent cannot affect the choices available to another (see [6] and [20] for a discussion of this property).*
- *For $\alpha \in Ags$, $\sim_\alpha$ is an equivalence relation on the set of* indices*, meant to express the epistemic indistinguishability relation for α, that satisfies the following conditions:*
  - ($\mathtt{OAC}$) Own action condition*: for index $\langle m, h\rangle$, $\langle m, h\rangle \sim_\alpha \langle m, h'\rangle$ for every $h' \in \mathbf{Choice}^m_\alpha(h)$.*
  - ($\mathtt{Unif-H}$) Uniformity of historical possibility*: if $\langle m, h\rangle \sim_\alpha \langle m', h'\rangle$, then, for $h_* \in H_m$, there exists $h'_* \in H_{m'}$ such that $\langle m, h_*\rangle \sim_\alpha \langle m', h'_*\rangle$.*

  For $\alpha \in Ags$, two notions of α's information set *at $\langle m, h\rangle$ are defined: the set $\pi^\Box_\alpha[\langle m, h\rangle] := \{\langle m', h'\rangle; \exists h'' \in H_{m'} s.t. \langle m, h\rangle \sim_\alpha \langle m', h''\rangle\}$ is α's* ex ante *information set; and the set $\pi_\alpha[\langle m, h\rangle] := \{\langle m', h'\rangle; \langle m, h\rangle \sim_\alpha \langle m', h'\rangle\}$ is α's* ex interim *information set.*
- *τ is a function that assigns to each $\alpha \in Ags$ and index $\langle m, h\rangle$ a topology $\tau^{\langle m, h\rangle}_\alpha$ on $\pi^\Box_\alpha[\langle m, h\rangle]$. This is the* topology of α's intentionality at $\langle m, h\rangle$, *where each open set is interpreted as a p-d intention of α at $\langle m, h\rangle$. τ must satisfy the following conditions:*

- (CI) Consistency of intention*: for every non-empty* $U, V \in \tau_\alpha^{\langle m,h \rangle}$*,* $U \cap V \neq \emptyset$*. In other words, every non-empty* U *is* $\tau_\alpha^{\langle m,h \rangle}$*-dense.*
- (KI) Knowledge of intention*: for* $\alpha \in Ags$ *and index* $\langle m, h \rangle$*, if* $\pi_\alpha^\Box[\langle m, h \rangle] = \pi_\alpha^\Box[\langle m', h' \rangle]$*, then* $\tau_\alpha^{\langle m,h \rangle} = \tau_\alpha^{\langle m',h' \rangle}$*.*

An iebt*-model* $\mathcal{M}$*, then, results from adding a valuation function* $\mathcal{V}$ *to an* iebt*-frame, where* $\mathcal{V} : P \to 2^{I(M \times H)}$ *assigns to each atomic proposition a set of indices.*

Definition 5 (Evaluation rules for intentionality). *Let* $\mathcal{M}$ *be an* iebt*-model. The semantics on* $\mathcal{M}$ *for the formulas of* $\mathcal{L}_I$ *are recursively defined as follows:*

$$\begin{array}{lll}
\mathcal{M}, \langle m,h \rangle \models p & \text{iff} & \langle m,h \rangle \in \mathcal{V}(p) \\
\mathcal{M}, \langle m,h \rangle \models \neg\varphi & \text{iff} & \mathcal{M}, \langle m,h \rangle \not\models \varphi \\
\mathcal{M}, \langle m,h \rangle \models \varphi \wedge \psi & \text{iff} & \mathcal{M}, \langle m,h \rangle \models \varphi \text{ and } \mathcal{M}, \langle m,h \rangle \models \psi \\
\mathcal{M}, \langle m,h \rangle \models \Box\varphi & \text{iff} & \text{for } h' \in H_m, \mathcal{M}, \langle m,h' \rangle \models \varphi \\
\mathcal{M}, \langle m,h \rangle \models [\alpha]\varphi & \text{iff} & \text{for } h' \in \mathbf{Choice}_\alpha^m(h), \mathcal{M}, \langle m,h' \rangle \models \varphi \\
\mathcal{M}, \langle m,h \rangle \models K_\alpha\varphi & \text{iff} & \text{for } \langle m',h' \rangle \text{ s.t. } \langle m,h \rangle \sim_\alpha \langle m',h' \rangle, \\
 & & \mathcal{M}, \langle m',h' \rangle \models \varphi \\
\mathcal{M}, \langle m,h \rangle \models I_\alpha\varphi & \text{iff} & \text{there exists } U \in \tau_\alpha^{\langle m,h \rangle} \text{ s.t. } U \subseteq \|\varphi\|,
\end{array}$$

where $\|\varphi\|$ *denotes the set* $\{\langle m,h \rangle \in I(M \times H); \mathcal{M}, \langle m,h \rangle \models \varphi\}$*.*

Therefore, one says that at index $\langle m, h \rangle$ α p-d intended φ iff there exists $U \in \tau_\alpha^{\langle m,h \rangle}$ that supports φ. Following [2], we will say that at $\langle m, h \rangle$ α had *ex ante* knowledge of φ iff $\mathcal{M}, \langle m, h \rangle \models \Box K_\alpha \varphi$—that is, iff at $\langle m, h \rangle$ it was historically settled that the agent knew φ.

Discussion

The reader might be curious as to why we chose a topological semantics. There are two main reasons:

1. Inspired by [21]'s and [15, Chapter 5]'s ideas behind using neighborhood semantics for formalizing intentionality, we opted to represent intentions as special subsets of indices in *bt*-models.[5] However, unlike these two approaches, we do not agree with the idea that p-d intentions should not be closed under logical consequence—in the case of logically omniscient agents. Thus, we started considering the idea of topologies of intentions, and we found out that we could use topological notions like *open*, *closed*, and *dense* sets to qualitatively describe a relation between p-d intentions and intentional action—in terms of measurement, closeness, and consistency. In our formalism, p-d intentions are "close" to both actual and intentional action, in the sense that any p-d intention supporting an agent's action—something

[5] As pointed out by [27] in his lecture notes for a course on neighborhood semantics, "[s]ets paired with a distinguished collections of subsets are ubiquitous in many areas of mathematics."

that will be necessary for intentional action—must be consistent with all other p-d intentions at the moment of acting. This is the reason behind the requirement that each open set is dense.

2. Topological semantics generalize standard relational semantics, and, in the words of [26, Chapter 1, p. 2], "topological spaces are equipped with well-studied basic operators such as the interior and closure operators which, alone or in combination with each other, succinctly interpret different modalities, giving a better understanding of their axiomatic properties."

It is important to emphasize that, for $\alpha \in Ags$ and index $\langle m, h\rangle$, the topology $\tau_\alpha^{\langle m,h\rangle}$ is a topology on α's *ex ante* information set. This implies that our semantics satisfy what we call the *knowledge-to-intention property* and the *knowledge-of-intention property*:

– *Knowledge-to-intention property*: all p-d intentions are included in an agent's *ex ante* information set. To clarify, this property is reflected by the validity of the formula $\Box K_\alpha \varphi \to I_\alpha \varphi$. [10] stated that intentions should concern states of affairs that an agent considers to be epistemically possible. In other words, an agent would be irrational in intending a state of affairs that the agent itself has ruled out, knowing that it is impossible for such a state of affairs to happen. We agree with this claim: an agent cannot but intend everything already known to be settled, because it would be irrational to do otherwise.[6]
– *Knowledge-of-intention property*: at an index an agent α always knew *ex ante* its p-d intentions. To clarify, this property is reflected by the fact that the formulas $I_\alpha \varphi \to \Box K_\alpha I_\alpha \varphi$ and $\neg I_\alpha \varphi \to \Box K_\alpha \neg I_\alpha \varphi$ are valid. This is a desirable property in virtue of a relatively usual assumption of positive and negative introspection of one's own intentionality. According to [23], who formalize the relation between intentions and beliefs, agents have positive and negative introspection of their intentions with respect to their beliefs (see also [19] and [16]). This means that formulas corresponding to $I_\alpha \varphi \to B_\alpha I_\alpha \varphi$ and $\neg I_\alpha \varphi \to B_\alpha \neg I_\alpha \varphi$ are valid in their logics. [10] supported this claim, and takes it further so as to include positive and negative introspection of one's intentions with respect to *ex ante* knowledge, just as we do here.

In order to illustrate our semantics of intentionality, let us use *iebt*-models to present a formal analysis of a simple example.

Example 1. Recall the situation described at the beginning of this section, where you are a lawyer in the prosecution of a driver that ran over—and killed—a traffic officer. Consider the *iebt*-model $\mathcal{M}$ depicted in Figure 1.

Here, $Ags = \{driver\}$, and m_1 is a moment. There are two histories (h_1 and h_2) passing through m_1. At m_1 the choices of action available to *driver* are the following: R_1, standing for the choice of running over the traffic officer,

[6] Observe that α's *ex ante* knowledge, at a given index, is itself a p-d intention of α, as witnessed by the fact that, since $\tau_\alpha^{\langle m,h\rangle}$ is a topology on α's *ex ante* information set, such an information set must be an element of the topology.

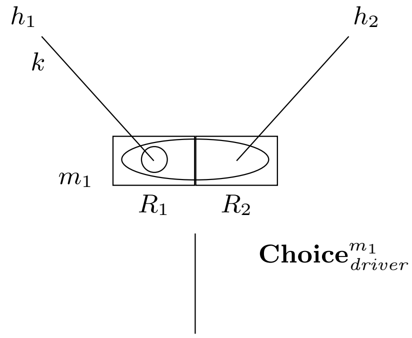

Fig. 1: Driver example

and R_2, standing for the choice of stopping the car. According to the choice performed, time moves on either into history h_1 or into history h_2. As implied by the statement of the example, h_1 is the *actual* history.

Here, *driver* is assumed to distinguish $\langle m_1, h_1\rangle$ from $\langle m_1, h_2\rangle$. Thus, at every index based on m_1 *driver* knew her choice of action. Moreover, *driver*'s *ex ante* information set at the actual index $\langle m_1, h_1\rangle$, denoted by $\pi^{\Box}_{driver}[\langle m_1, h_1\rangle]$, is the set $\{\langle m_1, h_1\rangle, \langle m_1, h_2\rangle\}$, which coincides with $\pi^{\Box}_{driver}[\langle m_1, h_2\rangle]$. As for *driver*'s intentionality, consider the topology $\tau^{\langle m_1,h_1\rangle}_{driver}$. Since $\pi^{\Box}_{driver}[\langle m_1, h_1\rangle] = \pi^{\Box}_{driver}[\langle m_1, h_2\rangle]$, frame condition ($\mathtt{KI}$) implies that $\tau^{\langle m_1,h_1\rangle}_{driver} = \tau^{\langle m_1,h_2\rangle}_{driver}$. The non-empty open sets of such a topology, then, are represented using circles and ellipses in the diagram. More precisely, $\tau^{\langle m_1,h_1\rangle}_{driver} = \left\{\emptyset, \pi^{\Box}_{driver}[\langle m_1, h_1\rangle], \{\langle m_1, h_1\rangle\}\right\}$.

Let k stand for the atomic proposition 'the traffic officer has been killed.' According to Definition 5, this atomic proposition and the formulas that are recursively built with it can be taken as true or false depending on the index of evaluation. For instance, $\mathcal{M}, \langle m_1, h_1\rangle \models K_{driver}[driver]k$: *at the actual index,* driver *knowingly killed the traffic officer*. As for formulas involving *driver*'s intentionality, let U denote the set $\{\langle m_1, h_1\rangle\}$. Then $U \in \tau^{\langle m_1,h_1\rangle}_{driver}$ and $U \subseteq \|k\|$. Thus, according to Definition 5, $\mathcal{M}, \langle m_1, h_1\rangle \models I_{driver}k$: *at the actual index* driver *had a p-d intention—or p-d intended—that the traffic officer was killed.* The same U attests to the fact that $\mathcal{M}, \langle m_1, h_1\rangle \models I_{driver}[driver]k$: *at the actual index* driver *had a p-d intention to see to it that the traffic officer was killed.* As such, for all practical purposes, the driver knowingly and intentionally killed the officer—which makes it reasonable for her to be blamed of second- or first-degree murder.

3 Logic-based Properties & Axiomatization

3.1 Properties

Let us review some properties of the logic that we have referred to as intentional epistemic stit theory, in terms of formulas that are either valid or invalid with respect to *iebt*-models. The logic-based properties of the modalities $\Box\varphi$ and $[\alpha]\varphi$ are the same in traditional stit theory. The properties of knowledge and its interplay with agency are the same as the ones addressed in [1]: K_α is an **S5** operator such that the formulas associated to frame conditions ($\mathtt{OAC}$)—$K_\alpha\varphi \to [\alpha]\varphi$—and ($\mathtt{Unif-H}$)—$\Diamond K_\alpha\varphi \to K_\alpha\Diamond\varphi$—are valid.

As for operator I_α, it turns out to be a **KD45** operator. The validity of the **KD45** schemata has the following consequences for our notion of intentionality, then:

- The validity of (K) implies that if at an index an agent p-d intended φ then the agent p-d intended all the logical consequences of φ. This property implies that our notion of intentionality is vulnerable to a particular version of the so-called *side-effect problem* (see [8], [12], and [10]). We do not agree with the claim that intention should not be closed under logical consequence. The reason is that we deal with idealized thinkers, who are logically omniscient and know, resp. believe, all the logical consequences of what they know, resp. believe. For idealized thinkers of this kind, then, we find it reasonable to assume that they will intend the logical consequences of whatever they intend.
- The validity of (D) $(I_\alpha\varphi \to \neg I_\alpha\neg\varphi)$ implies that if at an index an agent p-d intended φ then at that index the agent must not have p-d intended $\neg\varphi$. Most of the authors whose formalization of intention has been discussed in this section ([19], [12,23], [10], and [8]) support the idea that, at a specific point in time, future-directed intentions, p-d intentions, intentional actions, and intentions-with-which should be respectively consistent, and we agree with them.
- The validity of (4) $(I_\alpha\varphi \to I_\alpha I_\alpha\varphi)$ implies that if an agent p-d intended φ then at that index the agent p-d intended to p-d intend φ. Although this property is endorsed neither by [12] nor by [19], nor by [23], for instance, we consider it characteristic of p-d intentions, just as [9]: the *knowledge-of-intention property* implies that at an index an agent's p-d intentions are known *ex ante*, so that the *knowledge-to-intention property* implies that the agent cannot but have p-d intended to have those p-d intentions at that index.
- The validity of (5) $(\neg I_\alpha\varphi \to I_\alpha\neg I_\alpha\varphi)$ implies that if at an index an agent did not p-d intend φ then at that index the agent p-d intended to not p-d intend φ. Just as with the above property, out of all the works reviewed in this section, only [9] supports this property, as do we: the *knowledge-of-intention property* implies that at an index an agent's lack of a p-d intention toward the realization of φ is known *ex ante*, so that the *knowledge-to-intention*

property implies that the agent cannot but have p-d intended to not have had such a p-d intention at that index.

Furthermore, the validity, resp. invalidity, of the following formulas, with respect to the class of *iebt*-models, captures important properties of the interplay between the modalities of intentional epistemic stit theory:

1. (a) $\not\models I_\alpha\varphi \to I_\alpha[\alpha]\varphi$: it is not necessarily true that *if at an index an agent p-d intended φ, then at that index the agent p-d intended to see to it that φ.* This property refers to a distinction between intending that φ is the case and intending to be the material author of φ, on the other. For instance, suppose that I am a dictator displaying psychopathic traits. I have an intention toward the bombing of a neighboring country, but I do not intend for me to actually press any button deploying a bomb. Although some authors claim that the most primal notion of intending always refers to "intending to do" (see, for instance, [29] and [25]), we support the idea—consistent with [7]'s seminal thesis that future-directed intentions are elements in complex plans—that an agent can intend the realization of some state of affairs without intending to be the one realizing it.[7] Once again, a good example of this lies in "mastermind" agents that delegate actions to subordinates. The distinction between intending that φ is the case, on the one hand, and intending to actually see to it that φ, on the other, is all the more relevant in responsibility attribution: although my subordinate pilots were the ones deploying the bombs, it is me who should stand trial in The Hague. To illustrate this property, consider a variation of Example 1. Suppose that *driver* did not want to run over the traffic officer herself, but, still, she had a p-d intention that the traffic officer would get killed. A diagram of this situation is included in Figure 2.

   Observe that $\tau_{driver}^{\langle m_1,h_3\rangle} = \left\{\emptyset, \pi_{driver}^{\Box}\left[\langle m_1,h_3\rangle\right], \{\langle m_1,h_3\rangle\}\right\}$. Let $U = \{\langle m_1,h_3\rangle\}$. Then $U \subseteq \|k\|$. This means that $\mathcal{M},\langle m_1,h_3\rangle \models I_{driver}k$. However, there does not exist an open set included in $\|[driver]k\|$, which means that $\mathcal{M},\langle m_1,h_3\rangle \models \neg I_{driver}[driver]k$.

   (b) $\not\models I_\alpha[\alpha]\varphi \to [\alpha]\varphi \wedge I_\alpha[\alpha]\varphi$: it is not necessarily true that *if at an index an agent p-d intended to see to it that φ, then at that index the agent has intentionally seen to it that φ.* This property is related to the common assumption—following the ideas presented by [13]—that intending does not lead to intentionally doing. For instance, recall that I could have intended to start my car and still not have taken any action toward starting it. Therefore, this property is desirable for our interpretation of the conjunction $[\alpha]\varphi\wedge I_\alpha[\alpha]\varphi$—according to which $[\alpha]\varphi\wedge I_\alpha[\alpha]\varphi$ expresses that α has intentionally seen to it that φ. To illustrate this property,

[7] [15, Chapter 4, p. 163] explicitly states that there is a distinction between intending and intending to do. He writes: "[t]here are two different types of future-directed intentions: I can intend to perform a certain action, or I can intend to realize a certain state of affairs."

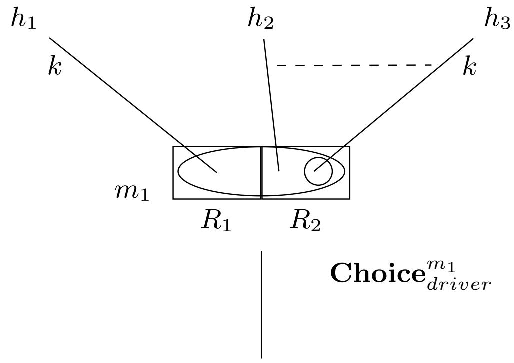

Fig. 2: Another driver example

consider Example 1: here, $\mathcal{M}, \langle m_1, h_2 \rangle \models I_\alpha[driver]k$ and $\mathcal{M}, \langle m_1, h_2 \rangle \models \neg[driver]k$: *at* $\langle m_1, h_2 \rangle$ driver *p-d intended to kill the traffic officer, but at such an index* driver *did not intentionally kill the traffic officer.*

(c) $\models I_\alpha[\alpha]\varphi \to I_\alpha\varphi$: *if at an index an agent p-d intended to see to it that* φ*, then at that index the agent p-d intended* φ*.* Since we interpret the conjunction $[\alpha]\varphi \wedge I_\alpha[\alpha]\varphi$ as α intentionally doing φ, then this property implies that intentional action implies intending in our framework. Thus, our notion of intentionality falls under a philosophical standpoint that [7] called the *Simple View*. The *Simple View* considers that, for an agent to intentionally do φ, the agent must also intend that φ is the case. Although [7] heavily objected to the *Simple View*, we find it appropriate for agents that are idealized thinkers. The validity of this formula follows from the validity of schema (T) for $[\alpha]$, Necessitation for I_α, and the validity of schema (K) for I_α.

(d) $\not\models [\alpha]\varphi \to I_\alpha\varphi$: it is not necessarily true that *if at an index an agent has seen to it that* φ*, then at that index the agent p-d intended* φ*.* This property, as well as its consequences in the present framework, reflects the desirable tenets that (i) not all actions follow a specific p-d intention, and that (ii) not all actions are intentional.

2. (a) $\not\models K_\alpha\varphi \to I_\alpha\varphi$: it is not necessarily true that *if at an index an agent knew* φ*, then at that index the agent p-d intended* φ*.* In light of the validity of the formulas associated to frame condition (`OAC`), $K_\alpha\varphi$ is logically equivalent to $K_\alpha[\alpha]\varphi$. To know φ, then, is to knowingly do φ. Therefore, this property—which can be reformulated as $\not\models K_\alpha[\alpha]\varphi \to I_\alpha\varphi$—reflects the desirable tenet that knowingly doing φ does not imply intending φ. An example of this situation is when someone else forced your hand. For instance, consider yet another variation of Example 1. Suppose, once again, that *driver* did not want to run over the traffic officer herself. By previously threatening to injure your family if you

refused to follow her instructions, *driver* forced you into taking *your* car and running over the traffic officer.

(b) $\not\models [\alpha]\varphi \wedge I_\alpha[\alpha]\varphi \to K_\alpha[\alpha]\varphi$: it is not necessarily true that *if at an index an agent has seen to it that* φ *and the agent p-d intended to see to it that* φ*, then at that index the agent has knowingly seen to it that* φ*.* This property entails that our framework allows us to model situations where an agent intentionally does φ without knowingly doing φ. A good example of the viability of such situations is when somebody intends to win a fair coin-flip and wins it by choosing heads—they could not have known that heads would make them win, but they still intentionally won.

(c) $\models \Box K_\alpha\varphi \to I_\alpha\varphi$: *if at an index an agent knew* φ ex ante*, then at that index the agent p-d intended* φ*.* The validity of $\Box K_\alpha\varphi \to I_\alpha\varphi$ reflects what we called the *knowledge-to-intention property* in the discussion right after Definition 5.

3. (a) $\models I_\alpha\varphi \to \Box K_\alpha I_\alpha\varphi$: *if at an index an agent p-d intended* φ*, then at that index the agent knew* ex ante *that it p-d intended* φ*.* Together with the formula in item 3b below, the validity of $I_\alpha\varphi \to \Box K_\alpha I_\alpha\varphi$ reflects what we called the *knowledge-of-intention property* in the discussion right after Definition 5, concerning the fact that at an index an agent must have known *ex ante* its p-d intentions. Now, such a property is connected to frame condition (`KI`) in Definition 4. Indeed, the formula $I_\alpha\varphi \to \Box K_\alpha I_\alpha\varphi$ defines (`KI`), such that an *iebt*-frame including τ that potentially violates (`KI`) satisfies (`KI`) iff $I_\alpha\varphi \to \Box K_\alpha I_\alpha\varphi$ is valid with respect to said frame. A proof of validity of the formula is straightforward.

   (b) $\models \neg I_\alpha\varphi \to \Box K_\alpha\neg I_\alpha\varphi$: *if at an index an agent did not p-d intend* φ*, then at that index the agent knew* ex ante *that it did not p-d intend* φ*.* In the proof system for intentional epistemic stit theory presented in Subsection 3.2, this formula can be derived using the one in item 3a above, so it is also valid.

Recall that at the introduction we mentioned that one of the main problems in philosophy of intention is the quest for unity in the three forms of intentionality (*future-directed intentions*, *intentional action*, and *intention-with-which*). The validity of the **KD45** schemata for I_α, then, coupled with the logic-based properties in item 1 above, somewhat settle where our interpretation of intentionality stands with respect to this problem. To clarify, first observe that we prioritize p-d intentions—which lie in the same category as future-directed intentions—and base on them both intentional action and intention-with-which. On the one hand, our framework's view on the relation between p-d intentions and intentional action is as follows: since we identify α's intentionally doing φ with the conjunction $[\alpha]\varphi \wedge I_\alpha[\alpha]\varphi$, then, for an action to count as intentional, an agent must have had a p-d intention of performing that action. In other words, at an index α has intentionally seen to it that φ only if α p-d intended to see to it that φ—that is, only if $I_\alpha[\alpha]\varphi$ holds. Therefore, the validity of the formula $I_\alpha[\alpha]\varphi \to I_\alpha\varphi$ (item 1c) implies that, for α to intentionally do φ, α must have p-d intended that φ would be the case. As mentioned before, this means that our treatment of intentionality falls under what [7] referred to as the *Simple View*. On the other hand, our framework's view on the relation between p-d intentions and intention-with-which is as follows: the validity of $\Box K_\alpha\varphi \to I_\alpha\varphi$ (item 2c), together with the validity of schema (K) for I_α, implies that the formula $(\Box K_\alpha(\varphi \to \psi) \wedge I_\alpha\varphi) \to I_\alpha\psi$ is valid. Therefore, if at an index α both knew *ex ante* that $[\alpha]\varphi \to \psi$ and has intentionally seen to it that φ, then the realization of ψ is an intention-with-which α has seen to it that φ—the formula $(\Box K_\alpha([\alpha]\varphi \to \psi) \wedge ([\alpha]\varphi \wedge I_\alpha[\alpha]\varphi)) \to I_\alpha\psi$ is valid.

3.2 Axiomatization

In this subsection we introduce a proof systems for our logic:

Definition 6 (Proof system for intentional epistemic stit theory). *Let Λ_I be the proof system defined by the following axioms and rules of inference:*

- (Axioms) *All classical tautologies from propositional logic; the* **S5** *schemata for $\Box$, $[\alpha]$, and K_α; the* **KD** *schemata for I_α; and the following schemata for the interactions between modalities:*

$$\Box\varphi \to [\alpha]\varphi \qquad (SET)$$

For $n \geq 1$ and pairwise different $\alpha_1, \ldots, \alpha_n$,

$$\bigwedge_{1\leq k\leq n} \Diamond[\alpha_i]\varphi_i \to \Diamond\left(\bigwedge_{1\leq k\leq n}[\alpha_i]\varphi_i\right) \qquad (IA)$$

$$K_\alpha\varphi \to [\alpha]\varphi \qquad (OAC)$$

$$\Diamond K_\alpha\varphi \to K_\alpha\Diamond\varphi \qquad (Unif-H)$$

$$\Box K_\alpha\varphi \to I_\alpha\varphi \qquad (InN)$$

$$I_\alpha\varphi \to \Box K_\alpha I_\alpha\varphi \qquad (KI)$$

- *(Rules of inference) Modus Ponens, Substitution, and Necessitation for all modal operators.*

Schemata (SET) and (IA) are standard in basic stit theory. Schema (OAC), resp. $(Unif-H)$, characterizes syntactically frame condition ($\mathtt{OAC}$), resp. frame condition ($\mathtt{Unif-H}$). Schema (InN)—where 'InN' stands for *intentional necessity*—characterizes syntactically what I called the *knowledge-to-intention property*. Schema (KI)—where 'KI' stands for *knowledge of intention*—characterizes syntactically the *knowledge-of-intention property*, as well as frame condition ($\mathtt{KI}$).

Remark 1. Schemata (4) and (5) for I_α, as well as schema $(\star)\neg I_\alpha\varphi \to \Box K_\alpha\neg I_\alpha\varphi$ and schema (Den) $\Diamond I_\alpha\varphi \to K_\alpha\langle I_\alpha\rangle\varphi$, are important Λ_I-theorems.

As for metalogic properties of intentional epistemic stit theory, the results of soundness and completeness for Λ_I are stated in the following theorem, whose proof is relegated to Appendix A:

Theorem 1. *The proof system Λ_I is sound and complete with respect to the class of* iebt-*models.*

□

The proof of Theorem 1 is the main technical contribution of this chapter. As for soundness, the proof is standard. As for completeness, the proof is a two-step process. First, we introduce a Kripke semantics for the logic—entirely based on relations on sets of possible worlds. In such a semantics, the formulas of $\mathcal{L}_{\mathsf{I}}$ are evaluated on Kripke-*ies*-models (Definition 9). We prove completeness of Λ_I with respect to the class of these structures, via the well-known technique of canonical models. Secondly, we provide a truth-preserving correspondence between Kripke-*ies*-models and a sub-class of *iebt*-models. Thus, completeness with respect to Kripke-*ies*-models yields completeness with respect to *iebt*-models. The second step implies associating a topological model to a Kripke model, such that both satisfy the same formulas at same indices. This is done via so-called *Alexandrov* spaces (Definition 7), with a technique inspired by [26] (see also [4] and [5]).

4 Conclusion

We want to conclude this work with a brief exploration of an important topic for future work: *using our theory of intentionality in the formalization of responsibility attribution.*

As first argued by [11] and afterwards by [15], one can classify the broad notion of responsibility in three categories: (1) *causal responsibility*, (2) *informational responsibility*, and (3) *motivational responsibility.* When talking causal responsibility, one wants to provide answers to the question "who is the material author of a given circumstance?" Informational responsibility concerns the question "did the author of a given circumstance behave consciously while performing the action that brought on such a circumstance?" Motivational responsibility, in turn, concerns the question "did the author of a given circumstance behave intentionally while performing the action that brought on such a circumstance?"

Observe, then, that we can model these three categories using intentional epistemic stit theory. In the spirit of [24], consider the following characterizations:

– *Causal Responsibility*: characterized by the formula $[\alpha]\varphi \wedge \Diamond\neg[\alpha]\varphi$, so that agent α was causally responsible for bringing about φ iff α saw to it that φ and it was possible for α to not see to it that φ,
– *Informational Responsibility*: characterized by the formula $K_\alpha[\alpha]\varphi \wedge K_\alpha\Diamond\neg\varphi$, so that α was informationally responsible for bringing about φ iff α knowingly saw to it that φ and α knew that it was possible to refrain from seeing to it that φ.
– *Motivational Responsibility*: characterized by the formula $I_\alpha[\alpha]\varphi \wedge K_\alpha\Diamond\neg\varphi$, so that α was motivationally responsible for bringing about φ iff α intentionally saw to it that φ and α knew that it was possible to refrain from seeing to it that φ.

Now, [11]'s initial motivation for categorizing the notion of responsibility was *mens rea*. As it turns out, our logic can also be used to formalize the modes of *mens rea*.

Suppose that φ stands for an illegal outcome, or a criminal offense. Thus, one can characterize the *mens rea* mode *purposefully*, for criminal agent α, with the formula $(K_\alpha[\alpha]\varphi \wedge I_\alpha[\alpha]\varphi)$. This formula holds at $\langle m, h\rangle$ iff at this index α was causally, informationally, and motivationally responsible for φ at $\langle m, h\rangle$. Similarly, one can characterize the *mens rea* mode *knowingly* with the formula $K_\alpha[\alpha]\varphi \wedge I_\alpha[\alpha]\varphi$. This formula holds at $\langle m, h\rangle$ iff at this index α was causally responsible, but not motivationally responsible, for φ at the index. As for the *mens rea* mode *recklessly*, one can characterize it with the formula $([\alpha]\theta \to \varphi) \wedge [\alpha]\theta \wedge K_\alpha \Diamond ([\alpha]\theta \to \varphi)$. This formula holds at $\langle m, h\rangle$ iff at this index (a) α causally brought about θ such that φ is a logical consequence of α's seeing to it that θ, and (b) α knew that it was possible that its bringing about θ could have implied φ. As for the *mens rea* mode *negligently*, one can characterize it with the formula $([\alpha]\theta \to \varphi) \wedge [\alpha]\theta \wedge \Box K_\beta ([\alpha]\theta \to \varphi)$, where β represents a legally *reasonable* agent. This formula holds at $\langle m, h\rangle$ iff at this index (a) α causally brought about θ such that φ is a logical consequence of α's seeing to it that θ, and (b) a reasonable agent β would have known *ex ante* about such a logical consequence. *Strict liability* offenses are charged and tried without appealing to any *mens rea* mental state. Typically, offenses of this kind can be divided in two main categories (see [22] and [18]): (1) minor infractions—such as speeding, overtime parking, or not signaling for a turn—for which the justification of reaching verdicts without requiring proof of *mens rea* is made on the grounds of regulatory expediency; and (2) serious crimes that pose a danger to society—such as statutory rape or felony murder—for which conviction without proof of *mens rea* is justified on the grounds of maximizing the deterrent effect of the penalty. For both categories, and if φ is a strict liability offense, one can characterize the mode *strict liability*—for criminal agent α—using α's causal-active responsibility for φ.

References

1. Abarca, A.I.R., Broersen, J.: A logic of objective and subjective oughts. In: European Conference on Logics in Artificial Intelligence. pp. 629–641. Springer (2019)
2. Abarca, A.I.R., Broersen, J.: Stit semantics for epistemic notions based on information disclosure in interactive settings. In: International Workshop on Dynamic Logic. pp. 171–189. Springer (2019)
3. Anscombe, G.E.M.: Intention. Cambridge: Harvard University Press (1963)
4. Baltag, A., Bezhanishvili, N., Ozgün, A., Smets, S.: The topology of full and weak belief (2014)
5. Baltag, A., Bezhanishvili, N., Özgün, A., Smets, S.: Justified belief and the topology of evidence. In: International Workshop on Logic, Language, Information, and Computation. pp. 83–103. Springer (2016)
6. Belnap, N., Perloff, M., Xu, M.: Facing the future: agents and choices in our indeterminist world. Oxford University Press (2001)
7. Bratman, M.: Two faces of intention. The Philosophical Review **93**(3), 375–405 (1984)
8. Bratman, M.: Intention, plans, and practical reason (1987)
9. Broersen, J.: A complete stit logic for knowledge and action, and some of its applications. In: International Workshop on Declarative Agent Languages and Technologies. pp. 47–59. Springer (2008)

10. Broersen, J.: A stit-logic for extensive form group strategies. In: 2009 IEEE/WIC/ACM International Joint Conference on Web Intelligence and Intelligent Agent Technology. vol. 3, pp. 484–487. IEEE (2009)
11. Broersen, J.: Deontic epistemic stit logic distinguishing modes of mens rea. Journal of Applied Logic **9**(2), 137–152 (2011)
12. Cohen, P.R., Levesque, H.J.: Intention is choice with commitment. Artificial intelligence **42**(2-3), 213–261 (1990)
13. Davidson, D.: Intending. In: Philosophy of history and action, pp. 41–60. Springer (1978)
14. Davidson, D.: Essays on Actions and Events. Oxford: Clarendon Press (1980)
15. Duijf, H.: Let's do it!: Collective responsibility, joint action, and participation. Ph.D. thesis, Utrecht University (2018)
16. Dunin-Keplicz, B., Verbrugge, R.: Collective intentions. Fundamenta Informaticae **51**(3), 271–295 (2002)
17. Engelking, R.: General topology (1989)
18. Green, S.P.: Six senses of strict liability: A plea for formalism. Appraising strict liability **1** (2005)
19. Herzig, A., Longin, D.: C&l intention revisited. KR **4**, 527–535 (2004)
20. Horty, J.F., Belnap, N.: The deliberative stit: A study of action, omission, ability, and obligation. Journal of philosophical logic **24**(6), 583–644 (1995)
21. Konolige, K., Pollack, M.E.: A representationalist theory of intention. In: IJCAI. vol. 93, pp. 390–395 (1993)
22. Larkin Jr, P.J.: Strict liability offenses, incarceration, and the cruel and unusual punishments clause. Harv. JL & Pub. Pol'y **37**, 1065 (2014)
23. Lorini, E., Herzig, A.: A logic of intention and attempt. Synthese **163**(1), 45–77 (2008)
24. Lorini, E., Longin, D., Mayor, E.: A logical analysis of responsibility attribution: emotions, individuals and collectives. Journal of Logic and Computation **24**(6), 1313–1339 (2014)
25. Moran, R., Stone, M.J.: Anscombe on expression of intention. In: New essays on the explanation of action, pp. 132–168. Springer (2009)
26. Özgün, A.: Evidence in epistemic logic: a topological perspective. Ph.D. thesis, Université de Lorraine (2017)
27. Pacuit, E.: Neighborhood semantics for modal logic: An introduction (2007), course Notes for ESSLLI
28. Setiya, K.: Intention. In: Zalta, E.N. (ed.) The Stanford Encyclopedia of Philosophy. Metaphysics Research Lab, Stanford University, Fall 2018 edn. (2018)
29. Thompson, M.: Life and action. Cambridge: Harvard University Press (2008)
30. Willard, S.: General topology. Courier Corporation (2004)

Appendix A Proofs of Soundness and Completeness

A.1 Soundness

Proposition 1. *The system Λ_I is sound with respect to the class of* iebt*-models.*

Proof. The proof of soundness is routine: the validity of (SET) and (IA) is standard from BST; the validity of (OAC) and $(Unif-H)$ is shown exactly as [1]; the validity of (InN) follows straightforwardly from Definitions 4 and 5; and the validity of (KI) follows from frame condition (KI).

A.2 Completeness

Definition 7 (*Alexandrov* spaces). *A topological space (X, τ) is said to be an* Alexandrov *space iff the intersection of any collection of open sets of X is an open set as well.*

Notice that a space is *Alexandrov* iff every point $x \in X$ has a $\subseteq$-smallest open set including it, namely the intersection of all the open sets around x.

Definition 8. *For a given frame (X, R) such that R is reflexive and transitive, a set $A \subseteq X$ is called upward-closed iff for $x \in A$, if $x \leq y$ for some $y \in X$, then $y \in A$ as well. For $x \in X$, $x \uparrow_R$ denotes the set $\{y \in X \mid xRy\}$, which is clearly upward closed.*

Remark 2. For a frame (X, R) such that R is reflexive and transitive, the set of all R-upward-closed sets forms an *Alexandrov* topology on X, which will be denoted by τ_R. For $x \in X$, the $\subseteq$-smallest open set including x is precisely $x \uparrow_R$. This implies that $\{x \uparrow_{\leq} \mid x \in X\}$ is a basis for the topology τ_R.

Definition 9 (Kripke-*ies*-frames & models).
A tuple

$$\left\langle W, Ags, R_{\Box}, Ags, \mathtt{Choice}, \{\approx_\alpha\}_{\alpha \in Ags}, \left\{R_\alpha^I\right\}_{\alpha \in Ags}\right\rangle$$

*is called a Kripke-*ies*-frame (where the acronym 'ies' stands for 'epistemic intentional stit') iff*

- *W is a set of possible worlds. $R_{\Box}$ is an equivalence relation over W. For $w \in W$, the class of w under $R_{\Box}$ is denoted by $\overline{w}$.*
- *$\mathtt{Choice}$ is a function that assigns to each $\alpha \in Ags$ and each $\Box$-class $\overline{w}$ a partition $\mathtt{Choice}_\alpha^{\overline{w}}$ of $\overline{w}$ given by an equivalence relation, denoted by $R_\alpha^{\overline{w}}$. $\mathtt{Choice}$ must satisfy the following constraint:*
  - $(\mathtt{IA})_{\mathtt{K}}$ *For $w \in W$, each function $s : Ags \to 2^{\overline{w}}$ that maps α to a member of $\mathtt{Choice}_\alpha^{\overline{w}}$ is such that $\bigcap_{\alpha \in Ags} s(\alpha) \neq \emptyset$.*

  For $\alpha \in Ags$, $w \in W$, and $v \in \overline{w}$, the class of v in the partition $\mathtt{Choice}_\alpha^{\overline{w}}$ is denoted by $\mathtt{Choice}_\alpha^{\overline{w}}(v)$.
- *For $\alpha \in Ags$, $\approx_\alpha$ is an (epistemic) equivalence relation on W that satisfies the following conditions:*
  - $(\mathtt{OAC})_{\mathtt{K}}$ *For $w \in W$ and $v \in \overline{w}$, $v \approx_\alpha u$ for every $u \in \mathtt{Choice}_\alpha^{\overline{w}}(v)$.*
  - $(\mathtt{Unif-H})_{\mathtt{K}}$ *Let $\alpha \in Ags$ and $v, u \in W$ such that $v \approx_\alpha u$. For $v' \in \overline{v}$, there exists $u' \in \overline{u}$ such that $v' \approx_\alpha u'$.*

  For $w, v \in W$ such that $w \approx_\alpha v$ and $L \subseteq \overline{w}$, L's epistemic cluster at $\overline{v}$ is the set $[\![L]\!]_\alpha^{\overline{v}} := \{u \in \overline{v};\ \text{there is } o \in L \ \text{ such that } \ o \approx_\alpha u\}$.
  For $\alpha \in Ags$ and $w \in W$, α's ex ante information set *at w is defined as $\pi_\alpha^{\Box}[w] := \{v; w \approx_\alpha \circ R_{\Box} v\}$, which by frame condition $(\mathtt{Unif-H})_{\mathtt{K}}$ coincides with the set $\{v; w R_{\Box} \circ \approx_\alpha v\}$. To clarify, $(\mathtt{Unif-H})_{\mathtt{K}}$ implies that $R_{\Box} \circ \approx_\alpha = \approx_\alpha \circ R_{\Box}$. Thus $\approx_\alpha \circ R_{\Box}$ is an equivalence relation such that $\pi_\alpha^{\Box}[w] = \pi_\alpha^{\Box}[v]$ for every $w, v \in W$ such that $w \approx_\alpha \circ R_{\Box} v$.*

– *For $\alpha \in Ags$, R^I_α is a serial, transitive, and euclidean relation on W such that $R^I_\alpha \subseteq \approx_\alpha \circ R_\Box$ and such that the following condition is satisfied:*
  - $(\mathtt{Den})_\mathtt{K}$ *For $v, u \in W$ such that $v \approx_\alpha \circ R_\Box u$, there exists $u' \in W$ such that $vR^I_\alpha u'$ and $uR^I_\alpha u'$.*

  For $\alpha \in Ags$, R^{I+}_α denotes the reflexive closure of R^I_α.

*A Kripke-*ies*-model $\mathcal{M}$ consists of the tuple that results from adding a valuation function $\mathcal{V}$ to a Kripke-*ies*-frame, where $\mathcal{V} : P \to 2^W$ assigns to each atomic proposition a set of worlds.*

Kripke-*ies*-models allow us to evaluate the formulas of $\mathcal{L}_\mathsf{I}$ with semantics that are *analogous* to the ones provided for *iebt*-frames. The semantics for the formulas of $\mathcal{L}_\mathsf{I}$ are given in the definition below.

Definition 10 (Evaluation rules on Kripke models). *Let $\mathcal{M}$ be a Kripke-*ies*-model. The semantics on $\mathcal{M}$ for the formulas of $\mathcal{L}_\mathsf{I}$ are defined recursively by the following truth conditions, evaluated at a given world w:*

$$\begin{array}{lll}
\mathcal{M}, w \models p & \textit{iff} & w \in \mathcal{V}(p) \\
\mathcal{M}, w \models \neg\varphi & \textit{iff} & \mathcal{M}, w \not\models \varphi \\
\mathcal{M}, w \models \varphi \wedge \psi & \textit{iff} & \mathcal{M}, w \models \varphi \textit{ and } \mathcal{M}, w \models \psi \\
\mathcal{M}, w \models \Box\varphi & \textit{iff} & \textit{for } v \in \overline{w}, \mathcal{M}, v \models \varphi \\
\mathcal{M}, w \models [\alpha]\varphi & \textit{iff} & \textit{for } v \in \mathtt{Choice}^{\overline{w}}_\alpha(w), \mathcal{M}, v \models \varphi \\
\mathcal{M}, w \models K_\alpha\varphi & \textit{iff} & \textit{for } v \textit{ s.t. } w \approx_\alpha v, \mathcal{M}, v \models \varphi \\
\mathcal{M}, w \models I_\alpha\varphi & \textit{iff} & \textit{there exists } x \in \pi^\Box_\alpha[w] \textit{ s.t. } x\uparrow_{R^{I+}_\alpha} \subseteq |\varphi|.
\end{array}$$

where I write $|\varphi|$ to refer to the set $\{w \in W; \mathcal{M}, w \models \varphi\}$. Satisfiability, validity on a frame, and general validity are defined as usual.

Definition 11 (Associated *iebt*-frame).
Let

$$\mathcal{F} = \left\langle W, Ags, R_\Box, \mathtt{Choice}, \{\approx_\alpha\}_{\alpha \in Ags}, \left\{R^I_\alpha\right\}_{\alpha \in Ags} \right\rangle$$

*be a Kripke-*ies*-frame. Then $\mathcal{F}^T := \left\langle M_W, \sqsubset, Ags, \mathbf{Choice}, \{\sim_\alpha\}_{\alpha \in Ags}, \tau \right\rangle$ is called the* iebt*-frame associated to $\mathcal{F}$ iff*

– *$M_W := W \cup \{\overline{w}; w \in W\} \cup \{W\}$, and $\sqsubset$ is a relation on M_W such that $\sqsubset$ is defined as the transitive closure of the union $\{(\overline{w}, v); w \in W$ and $v \in \overline{w}\} \cup \{(W, \overline{w}); w \in W\}$.*
  It is clear that $\sqsubset$ is a strict partial order on M_W that satisfies "no backward branching" *straightforwardly. Since the tuple $\langle M_W, \sqsubset\rangle$ is thus a tree, let us refer to the maximal $\sqsubset$-chains in M_W as* histories, *and let us denote by H_W the set of all histories of M_W. Observe that the definition of $\sqsubset$ yields that there is a bijective correspondence W and H_W. For $v \in W$, let h_v be the history $\{W, \overline{v}, v\}$. For $o \in W$, it is clear that $o \in h_v$ iff $o = v$. Therefore, each history in H_W can be identified using the world at its terminal node. Consequently, for $w \in W$, if $H_{\overline{w}}$ denotes the set of histories passing through $\overline{w}$, then $H_{\overline{w}} = \{h_v; v \in \overline{w}\}$—since $\overline{w} \in h_v$ iff $v \in \overline{w}$. Observe, then, that $H_W = \{h_v; v \in W\}$.*

– *For $B \in 2^W$, let B^T denote the set $\{h_v;\ v \in B\}$. With such a terminology, we define* **Choice** *as a function on $Ags \times M_W$ given by the rules:*
  - *For $\alpha \in Ags$ and $v \in W$,* $\mathbf{Choice}(\alpha, v) = \{\{h_v\}\}$.
  - *For $\alpha \in Ags$ and $w \in W$,* $\mathbf{Choice}(\alpha, \overline{w}) = \left\{C_\alpha^T; C_\alpha \in \mathtt{Choice}_\alpha^{\overline{w}}\right\}$.
  - *For $\alpha \in Ags$,* $\mathbf{Choice}(\alpha, W) = \{H_W\}$.

  To keep notation consistent, the sets of the form $\mathbf{Choice}(\alpha, \overline{w})$ *will be denoted by* $\mathbf{Choice}_\alpha^{\overline{w}}$*. The choice-cell of a given h_v in* $\mathbf{Choice}_\alpha^{\overline{w}}$ *is denoted by* $\mathbf{Choice}_\alpha^{\overline{w}}(h_v)$.
– *For $\alpha \in Ags$, $\sim_\alpha$ is a relation on $I\left(M_W \times H_W\right)$ defined as follows:*

$$\begin{aligned}\sim_\alpha := &\left\{\left(\langle \overline{w}, h_v\rangle, \langle \overline{w'}, h_{v'}\rangle\right); w, w' \in W \text{ and } v \approx_\alpha v'\right\} \cup \\ &\left\{\left(\langle z, h_z\rangle, \langle z, h_z\rangle\right); z \in W\right\} \cup \\ &\left\{\left(\langle W, h_v\rangle, \langle W, h_{v'}\rangle\right); v, v' \in W\right\}.\end{aligned}$$

  It is clear that this definition entails that $\sim_\alpha$ is an equivalence relation for every $\alpha \in Ags$ and that, for $w \in W$ and $L \in \mathtt{Choice}_\alpha^{\overline{w}}$, $v \in [\![L]\!]_\alpha^{\overline{w}}$ iff $h_v \in \left[L^T\right]_\alpha^{\overline{w}}$.
– *τ is a function defined as follows:*
  - *For $\alpha \in Ags$ and $z \in W$,* $\tau_\alpha^{\langle z, h_z\rangle} = \left\{\emptyset, \pi_\alpha^\Box\left[\langle z, h_z\rangle\right]\right\}$
  - *For $\alpha \in Ags$, we first define a relation R_α^{IT} on $\left\{\langle \overline{w}, h_v\rangle; w \in W \text{ and } v \in \overline{w}\right\}$ by the rule: $\langle \overline{w}, h_v\rangle R_\alpha^{IT} \left\langle \overline{w'}, h_{v'}\right\rangle$ iff $v R_\alpha^I v'$. For $\alpha \in Ags$, $w \in W$, and $v \in \overline{w}$, then, we define $\tau_\alpha^{\langle \overline{w}, h_v\rangle}$ as the subspace topology of $\tau_{R_\alpha^{IT+}}$ on $\pi_\alpha^\Box\left[\langle \overline{w}, h_v\rangle\right]$.[8] Observe that, for $\alpha \in Ags$, $w \in W$, and $v \in \overline{w}$, $\pi_\alpha^\Box\left[\langle \overline{w}, h_v\rangle\right] = \left\{\left\langle \overline{v'}, h_{v'}\right\rangle; v' \in \pi_\alpha^\Box[v]\right\}$. Thus, the fact that $R_\alpha^I \subseteq \approx_\alpha \circ R_\Box$ implies that, for $\langle \overline{x}, h_x\rangle \in \pi_\alpha^\Box\left[\langle \overline{w}, h_v\rangle\right]$, $\langle \overline{x}, h_x\rangle \uparrow_{R_\alpha^{IT+}} \subseteq \pi_\alpha^\Box\left[\langle \overline{w}, h_v\rangle\right]$, so that $\pi_\alpha^\Box\left[\langle \overline{w}, h_v\rangle\right]$ is open in $\tau_{R_\alpha^{IT+}}$.*
  - *For $\alpha \in Ags$ and $v \in W$,* $\tau_\alpha^{\langle W, h_v\rangle} = \left\{\emptyset, \pi_\alpha^\Box\left[\langle W, h_v\rangle\right]\right\}$.

Proposition 2. *Let $\mathcal{F}$ be a Kripke-*ies*-frame. Then $\mathcal{F}^T$ is an* iebt*-frame.*

Proof. It amounts to showing that $\sqsubset$ is a strict partial order that satisfies no backward branching, that **Choice** is a function that satisfies frame conditions $(\mathtt{NC})$ and $(\mathtt{IA})$, that $\{\sim_\alpha\}_{\alpha \in Ags}$ is such that $\approx_\alpha$ is an equivalence relation for every $\alpha \in Ags$ and frame conditions $(\mathtt{OAC})$ and $(\mathtt{Unif}-\mathtt{H})$ are met, and that τ is a function that meets the requirements of Definition 4.

– As mentioned in Definition 11, it is straightforward to show that $\sqsubset$ is a strict partial order that satisfies no backward branching. It is also clear from Definition 11 that $\sim_\alpha$ is an equivalence relation for every $\alpha \in Ags$.
– $(\mathtt{NC})$ is vacuously validated at moment W. It is validated in moments of the form $\overline{w}$ $(w \in W)$, since two different histories never intersect in a moment later than $\overline{w}$. Finally, it is also validated in moments of the form v such that $v \in W$ (since there are no moments above v).

[8] For $A \subseteq X$ and τ a topology on X, the subspace topology of τ on A is the family $\{U \cap A\ | U \in \tau\}$.

– For (IA), we reason by cases:
  (a) At moment W, (IA) is validated straightforwardly, since $\mathbf{Choice}(\alpha, W) = \{H\}$ for each $\alpha \in Ags$.
  (b) For a moment of the form $\overline{w}$ (with $w \in W$), let s be a function that assigns to each agent α a member of $\mathbf{Choice}_\alpha^{\overline{w}} = \left\{(C_\alpha)^T; C_\alpha \in \mathtt{Choice}_\alpha^{\overline{w}}\right\}$. Let $s_k : Ags \to \bigcup_{\alpha \in Ags} \mathtt{Choice}_\alpha^{\overline{w}}$ be a function such that $s_k(\alpha) = C_\alpha$ iff $s(\alpha) = (C_\alpha)^T$. Since $\mathcal{M}$ satisfies condition $(\mathtt{IA})_\mathtt{K}$, then $\bigcap_{\alpha \in Ags} s_k(\alpha) \neq \emptyset$. Take $v \in \bigcap_{\alpha \in Ags} s_k(\alpha)$. Then $v \in C_\alpha$ for every $\alpha \in Ags$. This implies that $h_v \in (C_\alpha)^T$ for every $\alpha \in Ags$, so $\bigcap_{\alpha \in Ags} s(\alpha) \neq \emptyset$.
  (c) At moments of the form v such that $v \in W$, if s is a function that assigns to each agent α a member of $\mathbf{Choice}(v, \alpha)$, s must be constant and $\bigcap_{\alpha \in Ags} s(\alpha) = \{h_v\}$.

– For (OAC), again we reason by cases:
  (a) Assume that $\langle \overline{w}, h_v \rangle \sim_\alpha \left\langle \overline{w'}, h_{v'} \right\rangle$ (for $w, w' \in W$). This means that $v \approx_\alpha v'$. We want to show that, for every $h_u \in \mathbf{Choice}_\alpha^{\overline{w}}$ such that $h_u \in \mathbf{Choice}_\alpha^{\overline{w}}(h_v)$, $\langle \overline{w}, h_u \rangle \sim_\alpha \left\langle \overline{w'}, h_{v'} \right\rangle$. Therefore, let $h_u \in \mathbf{Choice}_\alpha^{\overline{w}}(h_v)$. By definition, this means that $u \in \mathtt{Choice}_\alpha^{\overline{w}}(v)$. Since $\mathcal{M}$ satisfies condition $(\mathtt{OAC})_\mathtt{K}$, this last fact implies, with $v \approx_\alpha v'$, that $u \approx_\alpha v'$, which in turn yields that $\langle \overline{w}, h_u \rangle \sim_\alpha \left\langle \overline{w'}, h_v \right\rangle$.
  (b) For indices based on moments of the form v such that $v \in W$, (OAC) is met straightforwardly, since for h_v the choice-cell in $\mathbf{Choice}(\alpha, v)$ to which h_v belongs is just $\{h_v\}$.
  (c) For indices based on moment W, (OAC) is also met straightforwardly, since for every $\alpha \in Ags$ $\sim_\alpha$ is defined such that $\langle W, h_v \rangle \sim_\alpha \langle W, h_{v'} \rangle$ for every pair of histories $h_v, h_{v'}$ in H.

– For $(\mathtt{Unif}-\mathtt{H})$, again we reason by cases:
  (a) Assume that $\langle \overline{w}, h_v \rangle \sim_\alpha \left\langle \overline{w'}, h_{v'} \right\rangle$ (for $w, w' \in W$). This means that $v \in \overline{w}$, $v' \in \overline{w'}$, and $v \approx_\alpha v'$. Let $h_z \in H_{\overline{w}}$ (which means that $z \in \overline{w}$). We want to show that there exists $h \in H_{\overline{w'}}$ such that $\langle \overline{w}, h_z \rangle \sim_\alpha \left\langle \overline{w'}, h \right\rangle$. Condition $(Unif - H)_\mathtt{K}$ gives us that there exists $z' \in \overline{w'}$ such that $z \approx_\alpha z'$, which by definiton of $\sim_\alpha$ means that $\langle \overline{w}, h_z \rangle \sim_\alpha \left\langle \overline{w'}, h_{z'} \right\rangle$.
  (b) For indices based on moments of the form v such that $v \in W$, $\langle v, h_v \rangle$ is $\sim_\alpha$-related only to itself, so $(\mathtt{Unif}-\mathtt{H})$ is met straightforwardly.
  (c) For indices based on moment W, $(\mathtt{Unif}-\mathtt{H})$ is also met straightforwardly, since $\langle W, h_v \rangle \sim_\alpha \langle W, h_{v'} \rangle$ for every $v, v' \in W$.

– As for τ, it is clear that, for $\alpha \in Ags$ and index $\langle m, h \rangle$ either of the form $\langle z, h_z \rangle$ $(z \in W)$ or of the form $\langle W, h_v \rangle$ $(v \in W)$, $\tau_\alpha^{\langle m,h \rangle}$ is a topology on $\pi_\alpha^\Box[\langle m, h \rangle]$ that satisfies frame conditions (CI) and (KI).
  Assume, then, that $\langle m, h \rangle$ is of the form $\langle \overline{w}, h_v \rangle$ such that $v \in \overline{w}$. Let $\alpha \in Ags$. By Definition 11, $\tau_\alpha^{\langle \overline{w}, h_v \rangle}$ is the subspace topology of $\tau_{R_\alpha^{IT+}}$ on $\pi_\alpha^\Box[\langle \overline{w}, h_v \rangle]$. Thus, it is clear that $\tau_\alpha^{\langle \overline{w}, h_v \rangle}$ is a topology on $\pi_\alpha^\Box[\langle \overline{w}, h_v \rangle]$, so that τ straightforwardly satisfies (KI). Let us show that condition (CI) is also satisfied: let $U, V \in \tau_\alpha^{\langle \overline{w}, h_v \rangle}$ such that U and V are non-empty. Let $\langle \overline{u}, h_u \rangle \in U$ and $\langle \overline{x}, h_x \rangle \in V$. Definition 11 implies that $u \approx_\alpha \circ R_\Box x$. $\mathcal{F}$'s condition $(\mathtt{Den})_\mathtt{K}$

implies that there exists $u' \in W$ such that $uR^I_\alpha u'$ and $xR^I_\alpha u'$, which implies that $\langle \overline{u}, h_u \rangle \, R^{IT}_\alpha \, \langle \overline{u'}, h_{u'} \rangle$ and $\langle \overline{x}, h_x \rangle \, R^{IT}_\alpha \, \langle \overline{u'}, h_{u'} \rangle$, by definition of R^{IT}_α. This means that $\langle \overline{u'}, h_{u'} \rangle \in \langle \overline{u}, h_u \rangle \uparrow_{R^{IT+}_\alpha}$ and $\langle \overline{u'}, h_{u'} \rangle \in \langle \overline{x}, h_x \rangle \uparrow_{R^{IT+}_\alpha}$. Since $\pi^\Box_\alpha [\langle \overline{w}, h_v \rangle]$ is open in $\tau_{R^{IT+}_\alpha}$, we know that $\langle \overline{u}, h_u \rangle \uparrow_{R^{IT+}_\alpha} \subseteq U$ and that $\langle \overline{x}, h_x \rangle \uparrow_{R^{IT+}_\alpha} \subseteq V$. Thus, $\langle \overline{u'}, h \rangle \in U \cap V$, so that U and V are $\tau^{\langle \overline{w}, h_v \rangle}_\alpha$-dense.

Let $\mathcal{M}$ be a Kripke-*ies*-model with valuation function $\mathcal{V}$. The frame upon which $\mathcal{M}$ is based has an associated *iebt*-frame. If to the tuple of this *iebt*-frame one adds a valuation function $\mathcal{V}^t$ such that $\mathcal{V}^t(p) = \{\langle \overline{w}, h_w \rangle ; w \in \mathcal{V}(p)\}$, the resulting model is called the *iebt*-model associated to $\mathcal{M}$.

Proposition 3. *Let $\mathcal{M}$ be a Kripke-*ies*-model, and let $\mathcal{M}^T$ denote its associated *iebt*-model. For φ of $\mathcal{L}_\mathsf{I}$ and $w \in W$, $\mathcal{M}, w \models \varphi$ iff $\mathcal{M}^T, \langle \overline{w}, h_w \rangle \models \varphi$.*

Proof. We proceed by induction on φ. For the base case, take a propositional letter p and an arbitrary $w \in W$. Then $\mathcal{M}, w \models p$ iff $w \in \mathcal{V}(p)$ iff $\langle \overline{w}, h_w \rangle \in \mathcal{V}^T(p)$ iff $\mathcal{M}^T, \langle \overline{w}, h_w \rangle \models p$. The cases of Boolean connectives are standard, so let us deal with the modal operators. Let $w \in W$ and $\alpha \in Ags$.

– ($\Box$) $\mathcal{M}, w \models \Box\varphi$ iff for every $v \in \overline{w}$, $\mathcal{M}, v \models \varphi$, which by induction hypothesis happens iff $\mathcal{M}^T, \langle \overline{v}, h_v \rangle \models \varphi$ for every $v \in \overline{w}$, which happens iff $\mathcal{M}^T, \langle \overline{w}, h_w \rangle \models \Box\varphi$, since it is the case that $h_v \in H_{\overline{w}}$ iff $v \in \overline{w}$.
– ($[\alpha]$) $\mathcal{M}, w \models [\alpha]\varphi$ iff for every $v \in W$ such that $wR^{\overline{w}}_\alpha v$, $\mathcal{M}, v \models \varphi$, which by induction hypothesis happens iff $\mathcal{M}^T, \langle \overline{w}, h_v \rangle \models \varphi$ for every $h_v \in \mathbf{Choice}^{\overline{w}}_\alpha(h_w)$, which in turn happens iff $\mathcal{M}^T, \langle \overline{w}, h_w \rangle \models [\alpha]\varphi$.
– (K_α) $\mathcal{M}, w \models K_\alpha\varphi$ iff for every $v \in W$ such that $w \approx_\alpha v$, $\mathcal{M}, v \models \varphi$, which by induction hypothesis occurs iff $\mathcal{M}^T, \langle \overline{v}, h_v \rangle \models \varphi$ for every $h_v \in H$ such that $\langle \overline{w}, h_w \rangle \sim_\alpha \langle \overline{v}, h_v \rangle$, which happens iff $\mathcal{M}^T, \langle \overline{w}, h_w \rangle \models K_\alpha\varphi$.
– (I_α) First, observe that the induction hypothesis implies that $\|\varphi\| = \{\langle \overline{w}, h_w \rangle ; w \in |\varphi|\}$. Therefore, $\mathcal{M}, w \models I_\alpha\varphi$ iff there exists $x \in \pi^\Box_\alpha[w]$ such that $x \uparrow_{R^{I+}_\alpha} \subseteq |\varphi|$ iff $\langle \overline{x}, h_x \rangle \uparrow_{R^{IT+}_\alpha} \subseteq \|\varphi\|$ iff there exists $U \in \tau^{\langle \overline{w}, h_w \rangle}_\alpha$ such that $U \subseteq \|\varphi\|$ iff $\mathcal{M}^T, \langle \overline{w}, h_w \rangle \models I_\alpha\varphi$.

A.3 Canonical Kripke-*ies*-structure

We show that the proof system Λ_I is complete with respect to the class of Kripke-*ies*-models. For each Λ_I-consistent formula φ, we build a canonical structure from the syntax that satisfies φ.

Definition 12 (Canonical Structure). *The tuple*

$$\mathcal{M} = \left\langle W^{\Lambda_I}, R_\Box, \mathtt{Choice}, \{\approx_\alpha\}_{\alpha \in Ags}, \left\{R^I_\alpha\right\}_{\alpha \in Ags}, \mathcal{V} \right\rangle$$

is called a canonical structure for Λ_I iff

– $W^{\Lambda_I} = \{w; w$ *is a* Λ_I-MCS$\}$. *$R_\Box$ is a relation on W^{Λ_I} defined by ther rule: for $w, v \in W^{\Lambda_I}$, $wR_\Box v$ iff $\Box\varphi \in w \Rightarrow \varphi \in v$ for every φ of $\mathcal{L}_\mathsf{I}$. For $w \in W^{\Lambda_I}$, the set $\left\{v \in W^{\Lambda_I}; wR_\Box v\right\}$ is denoted by $\overline{w}$.*

– $\mathtt{Choice}$ *is a function that assigns to each* α *and* $\overline{w}$ *a subset of* $2^{\overline{w}}$*, denoted by* $\mathtt{Choice}^{\overline{w}}_{\alpha}$*, and defined as follows: let* $R^{\overline{w}}_{\alpha}$ *be a relation on* $\overline{w}$ *such that, for* $w, v \in W^{\Lambda_I}$*,* $wR^{\overline{w}}_{\alpha}v$ *iff* $[\alpha]\varphi \in w \Rightarrow \varphi \in v$ *for every* φ *of* $\mathcal{L}_{\mathsf{I}}$*; if* $\mathtt{Choice}^{\overline{w}}_{\alpha}(v) := \left\{u \in \overline{w}; vR^{\overline{w}}_{\alpha}u\right\}$*, then* $\mathtt{Choice}^{\overline{w}}_{\alpha} := \left\{\mathtt{Choice}^{\overline{w}}_{\alpha}(v); v \in \overline{w}\right\}$*.*
– *For* $\alpha \in Ags$*,* $\approx_{\alpha}$ *is an epistemic relation on* W^{Λ_I} *given by the rule: for* $w, v \in W^{\Lambda_I}$*,* $w \approx_{\alpha} v$ *iff* $K_{\alpha}\varphi \in w \Rightarrow \varphi \in v$ *for every* φ *of* $\mathcal{L}_{\mathsf{I}}$*.*
– *For* $\alpha \in Ags$*,* R^I_{α} *is a relation on* W^{Λ_I} *given by the rule: for* $w, v \in W^{\Lambda_I}$*,* $wR^I_{\alpha}v$ *iff* $I_{\alpha}\varphi \in w \Rightarrow \varphi \in v$ *for every* φ *of* $\mathcal{L}_{\mathsf{I}}$*.*
– $\mathcal{V}$ *is the canonical valuation, defined such that* $w \in \mathcal{V}(p)$ *iff* $p \in w$*.*

Proposition 4. *The canonical structure* $\mathcal{M}$ *for* Λ_I *is a Kripke*-ies-*model.*

Proof. We want to show that the tuple $\left\langle W^{\Lambda_I}, R_{\Box}, \mathtt{Choice}, \{\approx_{\alpha}\}_{\alpha \in Ags}, \left\{\mathtt{R}^I_{\alpha}\right\}_{\alpha \in Ags}\right\rangle$ is a Kripke-*ies*-frame, which amounts to showing that the tuple satisfies the items in the definition of Kripke-*ies*-frames (Definition 9).

– It is clear that $R_{\Box}$ is an equivalence relation, since Λ_I includes the **S5** axioms for $\Box$.
– Since Λ_I includes the **S5** schemata for $[\alpha]$ ($\alpha \in Ags$), $R^{\overline{w}}_{\alpha}$ is an equivalence relation for $\alpha \in Ags$ and $w \in W^{\Lambda_I}$. Moreover, since Λ_I includes schema (SET), $R^{\overline{w}}_{\alpha} \subseteq \overline{w} \times \overline{w}$ for every $w \in W^{\Lambda_I}$. Thus, $\mathtt{Choice}$ indeed assigns to each α and $\overline{w}$ a partition of $\overline{w}$.
To show that frame condition $(\mathtt{IA})_{\mathtt{K}}$ is satisfied, we first prove two intermediate results:
(a) For $w_* \in W^{\Lambda_I}$, $w \in \overline{w_*}$ iff $\{\Box\psi; \Box\psi \in w_*\} \subseteq w$. ($\Rightarrow$) Let $w \in \overline{w_*}$ (which means that $w_* R_{\Box} w$). Take φ of $\mathcal{L}_{\mathsf{KX}}$ such that $\Box\varphi \in w_*$. Since w_* is closed under *Modus Ponens*, axiom (4) for $\Box$ implies that $\Box\Box\varphi \in w_*$. By definition of $R_{\Box}$, $\Box\varphi \in w$. ($\Leftarrow$) Assume that $\{\Box\psi; \Box\psi \in w_*\} \subseteq w$. Take φ of $\mathcal{L}_{\mathsf{KX}}$ such that $\Box\varphi \in w_*$. By assumption, $\Box\varphi \in w$. Since w is closed under *Modus Ponens*, axiom (T) for $\Box$ implies that $\varphi \in w$. Thus, $w_* R_{\Box} w$ and $w \in \overline{w_*}$.
(b) If $w_* \in W^{\Lambda_I}$ and $s : Ags \to 2^{\overline{w_*}}$ maps α to a member of $\mathtt{Choice}^{\overline{w_*}}_{\alpha}$ such that $v_{s(\alpha)} \in s(\alpha)$, then $w \in s(\alpha)$ iff $\Delta_{s(\alpha)} = \left\{[\alpha]\psi; [\alpha]\psi \in v_{s(\alpha)}\right\} \subseteq w$. ($\Rightarrow$) Let $w \in s(\alpha)$ (which means that $v_{s(\alpha)} R_{\alpha} w$). Take φ of $\mathcal{L}_{\mathsf{KX}}$ such that $[\alpha]\varphi \in v_{s(\alpha)}$. Since $v_{s(\alpha)}$ is closed under *Modus Ponens*, schema (4) for $[\alpha]$ implies that $[\alpha][\alpha]\varphi \in v_{s(\alpha)}$. Therefore, by definition of R_{α}, $[\alpha]\varphi \in w$. ($\Leftarrow$) Assume that $\Delta_{s(\alpha)} = \left\{[\alpha]\psi; [\alpha]\psi \in v_{s(\alpha)}\right\} \subseteq w$. Take φ of $\mathcal{L}_{\mathsf{KX}}$ such that $[\alpha]\varphi \in v_{s(\alpha)}$. By assumption, $[\alpha]\varphi \in w$. Since w is closed under *Modus Ponens*, axiom (T) for $[\alpha]$ implies that $\varphi \in w$. Thus, $v_{s(\alpha)} R^{\overline{w_*}}_{\alpha} w$ and $w \in s(\alpha)$.
Next we show that, for $w_* \in W^{\Lambda_I}$ and $s : Ags \to 2^{\overline{w_*}}$ just as in item b above, $\bigcup_{\alpha \in Ags} \Delta_{s(\alpha)} \cup \{\Box\psi; \Box\psi \in w_*\}$ is Λ_I-consistent: first we show that $\bigcup_{\alpha \in Ags} \Delta_{s(\alpha)}$ is consistent. Suppose that this is not the case. Then there exists a set $\{\varphi_1, \ldots, \varphi_n\}$ of formulas of $\mathcal{L}_{\mathsf{KX}}$ such that $[\alpha_i]\varphi_i \in v_{s(\alpha_i)}$ for every $1 \leq i \leq n$ and

$$\vdash_{\Lambda_I} [\alpha_1]\varphi_1 \wedge \cdots \wedge [\alpha_n]\varphi_n \to \bot. \tag{1}$$

Without loss of generality, assume that $\alpha_i \neq \alpha_j$ for all $j \neq i$ such that $j, i \in \{1, \dots, n\}$—this assumption hinges on the fact that any stit operator distributes over conjunction. Notice that the fact that $[\alpha_i]\varphi_i \in v_{s(\alpha_i)}$ for every $1 \leq i \leq n$ implies that $\Diamond[\alpha_i]\varphi_i \in w_*$ for every $1 \leq i \leq n$. Since w_* is closed under conjunction, $\Diamond[\alpha_1]\varphi_1 \wedge \dots \wedge \Diamond[\alpha_n]\varphi_n \in w_*$.
Axiom (IA) then implies that

$$\vdash_{\Lambda_I} \Diamond[\alpha_1]\varphi_1 \wedge \dots \wedge \Diamond[\alpha_n]\varphi_n \to \Diamond\left([\alpha_1]\varphi_1 \wedge \dots \wedge [\alpha_n]\varphi_n\right). \tag{2}$$

Therefore, equations (2) and (1), imply that

$$\vdash_{\Lambda_I} \Diamond[\alpha_1]\varphi_1 \wedge \dots \wedge \Diamond[\alpha_n]\varphi_n \to \Diamond\bot. \tag{3}$$

But this is a contradiction, since $\Diamond[\alpha_1]\varphi_1 \wedge \dots \wedge \Diamond[\alpha_n]\varphi_n \in w_*$, and w_* is a Λ_I-MCS. Therefore, $\bigcup_{\alpha \in Ags} \Delta_{s(\alpha)}$ is consistent. Secondly, we show that the union $\bigcup_{\alpha \in Ags} \Delta_{s(\alpha)} \cup \{\Box\psi; \Box\psi \in w_*\}$ is also consistent. Suppose that this is not the case. Since $\bigcup_{\alpha \in Ags} \Delta_{s(\alpha)}$ is consistent, there must exist sets $\{\varphi_1, \dots, \varphi_n\}$ and $\{\theta_1, \dots, \theta_m\}$ of formulas of $\mathcal{L}_{\mathsf{KX}}$ such that $[\alpha_i]\varphi_i \in v_{s(\alpha_i)}$ for every $1 \leq i \leq n$, $\Box\theta_i \in w_*$ for every $1 \leq i \leq m$, and

$$\vdash_{\Lambda_I} [\alpha_1]\varphi_1 \wedge \dots \wedge [\alpha_n]\varphi_n \wedge \Box\theta_1 \wedge \dots \wedge \Box\theta_m \to \bot. \tag{4}$$

Let $\theta = \theta_1 \wedge \dots \wedge \theta_m$. Since $\Box$ distributes over conjunction, $\vdash_{\Lambda_I} \Box\theta \leftrightarrow \Box\theta_1 \wedge \dots \wedge \Box\theta_m$, where it is important to mention that, since w_* is a Λ_I-MCS closed under logical equivalence, $\Box\theta \in w_*$. Thus, (4) implies that

$$\vdash_{\Lambda_I} \left([\alpha_1]\varphi_1 \wedge \dots \wedge [\alpha_n]\varphi_n\right) \to \neg\Box\theta. \tag{5}$$

Once again, assume without loss of generality that $\alpha_i \neq \alpha_j$ for all $j \neq i$ such that $j, i \in \{1, \dots, n\}$. By an argument analogous to the one used to show that $\bigcup_{\alpha \in Ags} \Delta_{s(\alpha)}$ is consistent, (5) implies that

$$\vdash_{\Lambda_I} \Diamond[\alpha_1]\varphi_1 \wedge \dots \wedge \Diamond[\alpha_n]\varphi_n \to \Diamond\neg\Box\theta. \tag{6}$$

This entails that $\Diamond\neg\Box\theta \in w_*$, but this is a contradiction, since the fact that $\Box\theta \in w_*$ implies with axiom (4) for $\Box$ that $\Box\Box\theta \in w_*$. Now, let u_* be the Λ_I-MCS that includes $\bigcup_{\alpha \in Ags} \Delta_{s(\alpha)} \cup \{\Box\psi; \Box\psi \in w_*\}$. By intermediate result a, it is clear that $u_* \in \overline{w_*}$. By intermediate result b, it is clear that $u_* \in s(\alpha)$ for every $\alpha \in Ags$. Therefore, we have shown that, for $w_* \in W$, each function $s : Ags \to 2^{\overline{w_*}}$ that maps α to a member of $\mathtt{Choice}_\alpha^{\overline{w_*}}$ is such that $\bigcap_{\alpha \in Ags} s(\alpha) \neq \emptyset$, which means that $\mathcal{M}$ satisfies $(\mathtt{IA})_{\mathtt{K}}$.

– Since the proof system Λ_I includes the **S5** axioms for K_α ($\alpha \in Ags$), $\approx_\alpha$ is an equivalence relation for $\alpha \in Ags$. We verify that $\mathcal{M}$ satisfies conditions $(\mathtt{OAC})_{\mathtt{K}}$ and $(\mathtt{Unif-H})_{\mathtt{K}}$.
For $(\mathtt{OAC})_{\mathtt{K}}$, let $w_* \in W^{\Lambda_I}$ and $\alpha \in Ags$. Assume that $v, u \in \overline{w_*}$ are such that $v \approx_\alpha u$. Let $v' \in \mathtt{Choice}_\alpha^{\overline{w_*}}(v)$. This means that $vR_\alpha v'$. We want to show that $v' \approx_\alpha u$, so let φ be a formula of $\mathcal{L}_{\mathsf{KO}}$ such that $K_\alpha\varphi \in v'$. By schema (4) for K_α, $K_\alpha K_\alpha\varphi \in v'$. Similarly, since all substitutions of axiom

(OAC) lie within v' and it is closed under *Modus Ponens*, $[\alpha]K_\alpha\varphi$ also lies in v'. Since $v' R_\alpha v$, this implies that $K_\alpha\varphi \in v$. Therefore, our assumption that $v \approx_\alpha u$ entails that $\varphi \in u$. With this, we have shown that the fact that $K_\alpha\varphi \in v'$ implies that $\varphi \in u$, which means that $v' \approx_\alpha u$.
For $(\mathtt{Unif-H})_\mathtt{K}$, let $v, u \in W^{\Lambda_I}$ such that $v \approx_\alpha u$. Take $v' \in \overline{v}$. We want to show that there exists $u' \in \overline{u}$ such that $v' \approx_\alpha u'$. We show that $u'' = \{\psi; K_\alpha\psi \in v'\} \cup \{\Box\psi; \Box\psi \in u\}$ is consistent. To do so, we first show that $\{\psi; K_\alpha\psi \in v'\}$ is consistent. Suppose for a contradiction that it is not consistent. Then there exists a set $\{\psi_1, \dots, \psi_n\}$ of formulas of $\mathcal{L}_{\mathsf{KX}}$ such that $K_\alpha\psi_i \in v'$ for every $1 \leq i \leq n$ and (a) $\vdash_{\Lambda_I} \psi_1 \wedge \cdots \wedge \psi_n \to \bot$. By Necessitation for K_α and its distributivity over conjunction, (a) implies that $\vdash_{\Lambda_I} K_\alpha\psi_1 \wedge \cdots \wedge K_\alpha\psi_n \to K_\alpha\bot$, but this is a contradiction, since v' is a Λ_I-MCS and it includes $K_\alpha\psi_1 \wedge \cdots \wedge K_\alpha\psi_n$. Next we show that $u'' = \{\psi; K_\alpha\psi \in v'\} \cup \{\Box\psi; \Box\psi \in u\}$ is also consistent. Suppose for a contradiction that it is not consistent. Since $\{\psi; K_\alpha\psi \in v'\}$ and $\{\Box\psi; \Box\psi \in u\}$ are consistent, there must exist sets $\{\varphi_1, \dots, \varphi_n\}$ and $\{\theta_1, \dots, \theta_m\}$ of formulas of $\mathcal{L}_{\mathsf{KX}}$ such that $K_\alpha\varphi_i \in v'$ for every $1 \leq i \leq n$, $\Box\theta_i \in w_2$ for every $1 \leq i \leq m$, and (b) $\vdash_{\Lambda_I} \varphi_1 \wedge \cdots \wedge \varphi_n \wedge \Box\theta_1 \wedge \cdots \wedge \Box\theta_m \to \bot$. Let $\theta = \theta_1 \wedge \cdots \wedge \theta_m$ and $\varphi = \varphi_1 \wedge \cdots \wedge \varphi_n$. Since $\Box$ distributes over conjunction, $\vdash_{\Lambda_I} \Box\theta \leftrightarrow \Box\theta_1 \wedge \cdots \wedge \Box\theta_m$, where it is important to mention that, since u is a Λ_I-MCS, then $\Box\theta \in u$ and $(\star)$ $\Box\Box\theta \in u$ as well. In this way, (b) implies that $\vdash_{\Lambda_I} \varphi \to \neg\Box\theta$ and thus that (c) $\vdash_{\Lambda_I} \Diamond\varphi \to \Diamond\neg\Box\theta$. Notice that the facts that $K_\alpha\varphi_i \in v'$ for every $1 \leq i \leq n$, that K_α distributes over conjunction, and that v' is a Λ_I-MCS imply that $K_\alpha\varphi \in v'$. The fact that $v' \in \overline{v}$ implies that $\Diamond K_\alpha\varphi \in v$, so that $(Unif-H)$ entails that $K_\alpha\Diamond\varphi \in v$. Now, this last inclusion implies, with our assumption that $v \approx_\alpha u$, that $\Diamond\varphi \in u$, which by (c) in turn yields that $\Diamond\neg\Box\theta \in u$, contradicting $(\star)$. Therefore, u'' is consistent. Let u' be the Λ_I-MCS that includes u''. It is clear from its construction that $u' \in \overline{u}$ and that $v' \approx_\alpha u'$. With this, we have shown that $\mathcal{M}$ satisfies condition $(\mathtt{Unif-H})_\mathtt{K}$.

- Since Λ_I includes the **KD45** schemata for I_α ($\alpha \in Ags$), then R^I_α is a serial, transitive, and euclidean relation on W, for $\alpha \in Ags$. Since Λ_I includes schema (InN), then $R^I_\alpha \subseteq \approx_\alpha \circ R_\Box$ for $\alpha \in Ags$.
  We now verify that frame condition $(\mathtt{Den})_\mathtt{K}$ is satisfied. Let $v, u \in W^{\Lambda_I}$ such that $v \approx_\alpha \circ R_\Box u$. This means that there exists w such that $v \in \overline{w}$ and $w \approx_\alpha u$. We want to show that there exists $uR^I_\alpha u'$ such that $vR^I_\alpha u'$. We show that $u'' = \{\psi; I_\alpha\psi \in v\} \cup \{\psi; I_\alpha\psi \in u\}$ is consistent. To do so, we first show that $\{\psi; I_\alpha\psi \in v\}$ is consistent. Suppose for a contradiction that it is not consistent. Then there exists a set $\{\psi_1, \dots, \psi_n\}$ of formulas of $\mathcal{L}_{\mathsf{KX}}$ such that $I_\alpha\psi_i \in v$ for every $1 \leq i \leq n$ and (a) $\vdash_{\Lambda_I} \psi_1 \wedge \cdots \wedge \psi_n \to \bot$. By Necessitation for I_α and its distributivity over conjunction, (a) implies that $\vdash_{\Lambda_I} I_\alpha\psi_1 \wedge \cdots \wedge I_\alpha\psi_n \to I_\alpha\bot$, but this is a contradiction, since v is a Λ_I-MCS and it includes $I_\alpha\psi_1 \wedge \cdots \wedge I_\alpha\psi_n$. Next we show that $u'' = \{\psi; I_\alpha\psi \in v\} \cup \{\psi; I_\alpha\psi \in u\}$ is also consistent. Suppose for a contradiction that it is not consistent. Since $\{\psi; I_\alpha\psi \in v'\}$ and $\{\psi; I_\alpha\psi \in u\}$ are consistent, there must exist sets $\{\varphi_1, \dots, \varphi_n\}$ and $\{\theta_1, \dots, \theta_m\}$ of formulas of $\mathcal{L}_{\mathsf{KX}}$ such

that $I_\alpha\varphi_i \in v$ for every $1 \le i \le n$, $I_\alpha\theta_i \in w_2$ for every $1 \le i \le m$, and (b) $\vdash_{\Lambda_I} \varphi_1\wedge\cdots\wedge\varphi_n\wedge\theta_1\wedge\cdots\wedge\Box\theta_m \to \bot$. Let $\theta = \theta_1\wedge\cdots\wedge\theta_m$ and $\varphi = \varphi_1\wedge\cdots\wedge\varphi_n$. Thus, (b) implies that $\vdash_{\Lambda_I} \varphi \to \neg\theta$ and thus that (c) $\vdash_{\Lambda_I} \langle I_\alpha\rangle\varphi \to \langle I_\alpha\rangle\neg\theta$. Notice that the facts that $I_\alpha\varphi_i \in v$ for every $1 \le i \le n$, that I_α distributes over conjunction, and that v is a Λ_I-MCS imply that $I_\alpha\varphi \in v$. Analogously, one has that $(\star)$ $I_\alpha\theta \in u$. The fact that $v \in \overline{w}$ implies that $\Diamond I_\alpha\varphi \in w$, so that (Den) entails that $K_\alpha\langle I_\alpha\rangle\varphi \in w$. Now, this last inclusion implies, with the fact that $w \approx_\alpha u$, that $\langle I_\alpha\rangle\varphi \in u$, which by (c) in turn yields that $\langle I_\alpha\rangle\neg\theta \in u$, contradicting $(\star)$. Therefore, u'' is consistent. Let u' be the Λ_I-MCS that includes u''. It is clear from its construction that $uR^I_\alpha u'$ and that $vR^I_\alpha u'$. With this, we have shown that $\mathcal{M}$ satisfies $(\mathtt{Den})_{\mathtt{K}}$.

Lemma 1 (Existence for non-intentional operators). *Let $\mathcal{M}$ be the canonical Kripke*-ies-*model for Λ_I. Let $w \in W^{\Lambda_I}$. For φ of $\mathcal{L}_{\mathsf{I}}$, the following items hold:*

1. *$\Box\varphi \in w$ iff $\varphi \in v$ for every $v \in \overline{w}$.*
2. *$[\alpha]\varphi \in w$ iff $\varphi \in v$ for every $v \in \overline{w}$ such that $wR^{\overline{w}}_\alpha v$.*
3. *$K_\alpha\varphi \in w$ iff $\varphi \in v$ for every $v \in W^{\Lambda_I}$ such that $w \approx_\alpha v$.*

Proof. Let $w \in W^{\Lambda_I}$, and take φ of $\mathcal{L}_{\mathsf{I}}$. All items are shown in the same way. Let $\triangle \in \{\Box, [\alpha], K_\alpha\}$, and let $R_\triangle$ stand for the relation upon which the semantics of $\triangle\varphi$ is defined. We show that $\triangle\varphi \in w$ iff $\varphi \in v$ for every $v \in W^{\Lambda_I}$ such that $wR_\triangle v$.

($\Rightarrow$) Assume that $\triangle\varphi \in w$. Let $v \in W^{\Lambda_I}$ such that $wR_\triangle v$. The definition of $R_\triangle$ straightforwardly gives that $\varphi \in v$.

($\Leftarrow$) We work by contraposition. Assume that $\triangle\varphi \notin w$. We show that there is a world v in W^{Λ_I} such that $wR_\triangle v$ and such that φ does not lie within it. For this, let $v' = \{\psi; \triangle\psi \in w\}$, which is shown to be consistent as follows: suppose for a contradiction that v' is not consistent; then there exists a set $\{\psi_1, \ldots, \psi_n\}$ of formulas of $\mathcal{L}_{\mathsf{KX}}$ such that $\{\psi_1, \ldots, \psi_n\} \subseteq v'$ and (a) $\vdash_{\Lambda_I} \psi_1 \wedge \cdots \wedge \psi_n \to \bot$; now, the fact that $\{\psi_1, \ldots, \psi_n\} \subseteq v'$ means that $\triangle\psi_i \in w$ for every $1 \le i \le n$; Necessitation for $\triangle$ and its distributivity over conjunction yield that (a) implies that $\vdash_{\Lambda_I} \triangle\psi_1\wedge\cdots\wedge\triangle\psi_n \to \triangle\bot$, but this is a contradiction, since w is a Λ_I-MCS which includes $\triangle\psi_1 \wedge \cdots \wedge \triangle\psi_n$. Now, we define $v' := v' \cup \{\neg\varphi\}$, and we show that it is also consistent: suppose for a contradiction that it is not consistent; since v' is consistent, there exists a set $\{\psi_1, \ldots, \psi_n\}$ of formulas of $\mathcal{L}_{\mathsf{KX}}$ such that $\{\psi_1, \ldots, \psi_n\} \subseteq v'$ and $\vdash_{\Lambda_I} \psi_1 \wedge \cdots \wedge \psi_n \wedge \neg\varphi \to \bot$, which then implies that (b) $\vdash_{\Lambda_I} \psi_1 \wedge \cdots \wedge \psi_n \to \varphi$; By Necessitation for $\triangle$ and its distributivity over conjunction, (b) implies that $\vdash_{\Lambda_I} \triangle\psi_1 \wedge \cdots \wedge \triangle\psi_n \to \triangle\varphi$; but, since w is a Λ_I-MCS, then $\triangle\psi_1 \wedge \cdots \wedge \triangle\psi_n \in w$, so that (b) and the fact that w is closed under *Modus Ponens* entail that $\triangle\varphi \in w$, contradicting the initial assumption that $\triangle\varphi \notin w$. Let v be the Λ_I-MCS that includes v''. It is clear from its construction that $\varphi \notin v$ and that $wR_\triangle v$, by definition of $R_\triangle$.

Lemma 2 (Truth Lemma). *Let $\mathcal{M}$ be the canonical Kripke*-ies-*model for Λ_I. For φ of $\mathcal{L}_{\mathsf{I}}$ and $w \in W^{\Lambda_I}$, $\mathcal{M}, w \models \varphi$ iff $\varphi \in w$.*

Proof. We proceed by induction on φ. The cases of Boolean connectives are standard. For formulas involving $\Box$, $[\alpha]$, and K_α, both directions follow straightforwardly from Lemma 1 (items 1, 2, and 3, respectively). As for I_α, we have the following arguments:

- ("I_α")
  ($\Rightarrow$) We work by contraposition. Suppose that $I_\alpha\varphi \notin w$. Take $x \in \pi_\alpha^\Box[w]$. The assumption that $\neg I_\alpha\varphi \in w$ implies, by schema (KI) and closure of w under *Modus Ponens*, that $\Box K_\alpha \neg I_\alpha\varphi \in w$. Since $x \in \pi_\alpha^\Box[w]$, this implies that $\neg I_\alpha\varphi \in x$. By an argument analogous to the one used in Proposition 4 to show that the canonical model satisfies $(\mathtt{Den})_\mathtt{K}$, the set $\{\psi; I_\alpha\psi \in x\}$ is consistent. Next, observe that $\{\psi; I_\alpha\psi \in w\} \cup \{\neg\varphi\}$ is consistent. Suppose it is not consistent. Since $\{\psi; I_\alpha\psi \in w\}$ is consistent, there must exist a set $\{\varphi_1, \ldots, \varphi_n\}$ such that $I_\alpha\varphi_i \in w$ for every $1 \leq i \leq n$ and $\vdash_{\Lambda_I} (\varphi_1 \wedge \cdots \wedge \varphi_n) \wedge \neg\varphi \to \bot$ Now, this Λ_I-theorem implies that $\vdash_{\Lambda_I} (\varphi_1 \wedge \cdots \wedge \varphi_n) \to \varphi$. By Necessitation of I_α, its schema (K), and its distributivity over conjunction, one then has that $(\star) \vdash_{\Lambda_I} (I_\alpha\varphi_1 \wedge \cdots \wedge I_\alpha\varphi_n) \to I_\alpha\varphi$. Now, closure of w under conjunction then implies that $\left(\bigwedge_{1 \leq i \leq n} I_\alpha\varphi_i\right) \in x$, so that the antecedent in Λ_I-theorem $(\star)$ lies in x. Closure of x under *Modus Ponens* then implies that $I_\alpha\varphi \in x$, but this contradicts the previously shown fact that $I_\alpha\varphi \notin x$. Therefore, $\{\psi; I_\alpha\psi \in w\} \cup \{\neg\varphi\}$ is in fact consistent. Let u be the Λ_I-MCS that includes $\{\psi; I_\alpha\psi \in w\} \cup \{\neg\varphi\}$. It is clear from construction that $xR_\alpha^I u$, so that $u \in x \uparrow_{R_\alpha^+}$. It also clear from construction that $\neg\varphi \in x$, so that the induction hypothesis yields that $\mathcal{M}, x \models \neg\varphi$. Thus, x is such that $x \in \pi_\alpha^\Box[w]$ and such that $x \uparrow_{R_\alpha^+} \not\subseteq |\varphi|$, which implies that $\mathcal{M}, w \not\models \varphi$.
  ($\Leftarrow$) Assume that $I_\alpha\varphi \in w$. Suppose for a contradiction that $\mathcal{M}, w \not\models I_\alpha\varphi$. This means that, for $x \in \pi_\alpha^\Box[w]$, there exists y such that $xR_\alpha^{I+}y$ and $\mathcal{M}, y \not\models \varphi$. Now, we have two cases. If for every $x \in \pi_\alpha^\Box[w]$ the y such that $xR_\alpha^{I+}y$ and $\mathcal{M}, y \not\models \varphi$ is actually x itself, then $\mathcal{M}, x \not\models \varphi$ for every $x \in \pi_\alpha^\Box[w]$. By induction hypothesis, this implies that $\neg\varphi \in x$ for every $x \in \pi_\alpha^\Box[w]$, which, by items 1 and 3 of Lemma 1, implies that $\Box K_\alpha \neg\varphi \in x$ for every $x \in \pi_\alpha^\Box[w]$. In particular, $\Box K_\alpha \neg\varphi \in w$. Schema (InN) and closure of w under *Modus Ponens* then imply that $I\neg\varphi \in w$, but this is a contradiction, since the fact that $I_\alpha\varphi \in w$, with schema (D) for I_α and closure of w under *Modus Ponens*, implies that $\neg I\neg\varphi \in w$. The other case is that there exist $x, y \in \pi_\alpha^\Box[w]$ such that $xR_\alpha^{I+}y$, $\mathcal{M}, y \not\models \varphi$, and $y \neq x$. By induction hypothesis, $\varphi \notin y$. Since $xR_\alpha^{I+}y$ and $y \neq x$, then $xR_\alpha^I y$, so the definition of R_α^I implies that $I_\alpha\varphi \notin x$. As such, $\neg I_\alpha\varphi \in x$, which, by schema (KI) and closure of x under *Modus Ponens*, implies that $\Box K_\alpha \neg I_\alpha\varphi \in x$. Since $x \in \pi_\alpha^\Box[w]$, this implies that $\neg I_\alpha\varphi \in w$, but this is a contradiction to the initial assumption.

Theorem 2 (Completeness w.r.t. Kripke-*ies*-models). *The proof system Λ_I is complete with respect to the class of Kripke-*ies*-models.*

Proof. Let φ be a Λ_I-consistent formula of $\mathcal{L}_\mathsf{I}$. Let w be the Λ_I-MCS including φ. The canonical Kripke-*ies*-model $\mathcal{M}$ for Λ_I is such that $\mathcal{M}, w \models \varphi$.

Back to branching-time models

Theorem 3 (Completeness w.r.t. *iebt*-models). *The proof system Λ_I is complete with respect to the class of* iebt-*models.*

Proof. Let φ be a Λ_I-consistent formula of $\mathcal{L}_\mathsf{I}$. Theorem 2 implies that there exists a Kripke-*ies*-model $\mathcal{M}$ and a world w in its domain such that $\mathcal{M}, w \models \varphi$. Proposition 3 then ensures that the *iebt*-model $\mathcal{M}^T$ associated to $\mathcal{M}$ is such that $\mathcal{M}^T, \langle \overline{w}, h_w \rangle \models \varphi$.

Therefore, the following result, appearing in the main body of the paper, has been shown:

Theorem 1. *The proof system Λ_I is sound and complete with respect to the class of* iebt-*models.*

$\square$