Cautious distributed belief

Dalí 2022

John Lindqvist Fernando R. Velázquez-Quesada Thomas Ågotnes

University of Bergen, Department of Information Science and Media Studies

Introduction

Outline:

- Introduction and motivation
- Definitions and example
- Some results
- Expressivity and bisimulation
- Conclusion

The idea

- Epistemic logic and distributed knowledge
- Distributed belief and inconsistent information
- Restricting the information combination?
- Evidence logics[1]: quantifying over maximal consistent evidence sets.
- "Cautious" distributed belief: quantifying over maximal consistent groups.

[1] Johan van Benthem and Eric Pacuit. "Dynamic Logics of Evidence-Based Beliefs". In: Studia Logica 99.1 (2011), pp. 61–92. ISSN: 00393215, 15728730. DOI: 10.1007/s11225-011-9347-x.

Related work

- Explicit beliefs and (consistent) distributed belief.[2]
- Distributed belief fusion.[3]

[2] Andreas Herzig et al. "A Logic of Explicit and Implicit Distributed Belief". In: ECAI 2020 - 24th European Conference on Artificial Intelligence. Vol. 325. Frontiers in Artificial Intelligence and Applications. IOS Press, 2020, pp. 753–760. DOI: 10.3233/FAIA200163.

[3] Churn-Jung Lian. "A conservative approach to distributed belief fusion". In: Proceedings of the Third International Conference on Information Fusion. Vol. 1. 2000, MOD4/3–MOD410 vol.1. DOI: 10.1109/IFIC.2000.862649.

Definitions

The models

- $A = \{a, b, ...\}$, $P = \{p, q, ...\}$.
- Belief model: tuple $\mathcal{M} = \langle W, R, v \rangle$,
  Pointed belief model: pair $(\mathcal{M}, s)$.
- Conjecture set: $C_a(s) := \{s' \in W \mid sR_a s'\}$ (for $a \in A$).
- Combined conjecture set: $C_G(s) := \bigcap_{a \in G} C_a(s)$ (for $G \subseteq A$).

Individual belief and standard distributed belief

$$\mathcal{M}, s \vDash B_a \varphi \quad \text{iff} \quad \forall s' \in C_a(s) \colon \mathcal{M}, s' \vDash \varphi,$$

$$\mathcal{M}, s \vDash D_G \varphi \quad \text{iff} \quad \forall s' \in C_G(s) \colon \mathcal{M}, s' \vDash \varphi,$$

Definitions

Maximal consistency

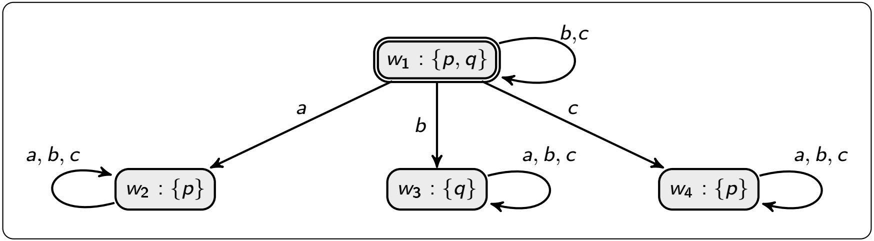
- Consistency (relative to a world): a group $G \subseteq A$ is consistent at the world s if and only if $C_G(s) \neq \varnothing$.
- Maximal consistency: a non-empty subgroup G' of $G \subseteq A$ is maximal consistent w.r.t G at s if and only if it is consistent and for every $G' \subset H \subseteq G$, H is inconsistent. In symbols: $G' \subseteq_s^{max} G$.

Definition 1 (Cautious distributed belief) *Let $(\mathcal{M}, s)$ be a pointed belief model with $\mathcal{M} = \langle W, R, v \rangle$; take $a \in A$ and a non-empty $G \subseteq A$. Then,*

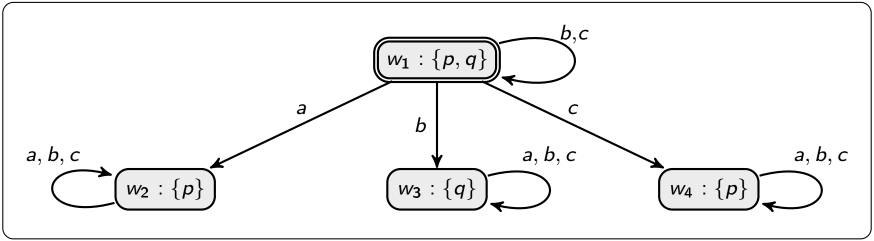
$$\mathcal{M}, s \models D_G^{\forall} \varphi \quad \text{iff} \quad \forall G' \subseteq_s^{max} G, \forall s' \in C_{G'}(s)\text{: } \mathcal{M}, s' \models \varphi. \qquad \blacktriangleleft$$

- $R_G^{\forall} ss'$ iff $\exists G' \subseteq_s^{max} G$ such that $s' \in C_{G'}(s)$.

$$\mathcal{M}, s \models D^{\forall}_{G}\varphi \text{ iff } \forall G' \subseteq^{max}_{s} G, \forall s' \in C_{G'}(s)\text{: } \mathcal{M}, s' \models \varphi.$$

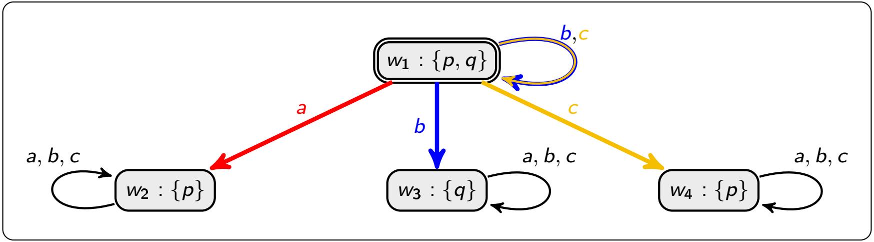


$\mathcal{M}, s \vDash D^{\forall}_{G}\varphi$ iff $\forall G' \subseteq^{max}_{s} G$, $\forall s' \in C_{G'}(s)$: $\mathcal{M}, s' \vDash \varphi$.



$G = \{a, b, c\}$:

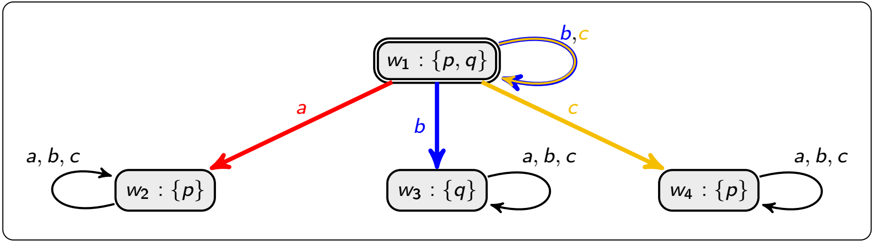
$\mathcal{M}, s \vDash D^{\forall}_{G}\, \varphi$ iff $\forall G' \subseteq^{max}_{s} G$, $\forall s' \in C_{G'}(s)$: $\mathcal{M}, s' \vDash \varphi$.



$G = \{a, b, c\}$:

$C_a(w_1) = \{w_2\}$, $C_b(w_1) = \{w_1, w_3\}$, $C_c(w_1) = \{w_1, w_4\}$

$\mathcal{M}, s \models D^{\forall}_{G}\varphi$ iff $\forall G' \subseteq^{max}_{s} G$, $\forall s' \in C_{G'}(s)$: $\mathcal{M}, s' \models \varphi$.

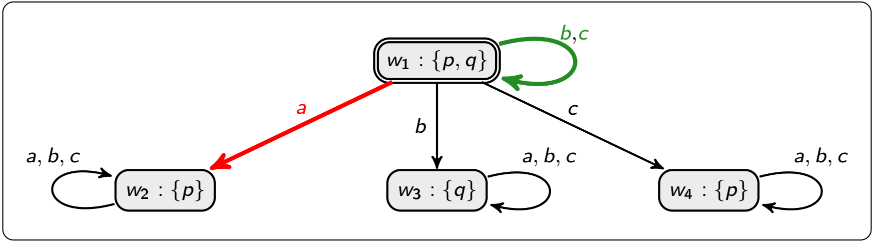


$G = \{a, b, c\}$:

$C_a(w_1) = \{w_2\}$, $C_b(w_1) = \{w_1, w_3\}$, $C_c(w_1) = \{w_1, w_4\}$

- $\mathcal{M}, w_1 \models B_a p \wedge B_a \neg q$, $\mathcal{M}, w_1 \models B_b q$, $\mathcal{M}, w_1 \models B_c p$.

$\mathcal{M}, s \models D^{\forall}_{G}\varphi$ iff $\forall G' \subseteq^{max}_{s} G, \forall s' \in C_{G'}(s)$: $\mathcal{M}, s' \models \varphi$.

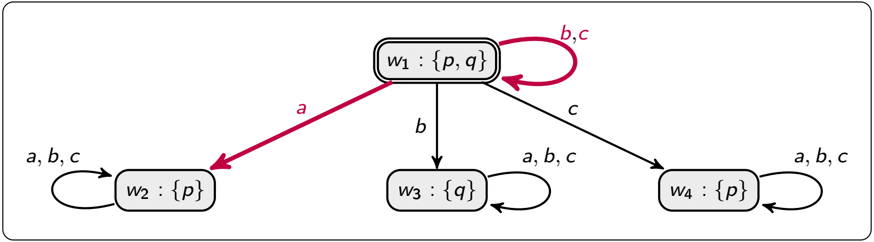


$G = \{a, b, c\}$:
$C_a(w_1) = \{w_2\}$, $C_b(w_1) = \{w_1, w_3\}$, $C_c(w_1) = \{w_1, w_4\}$

- $\mathcal{M}, w_1 \models B_a p \wedge B_a \neg q$, $\quad \mathcal{M}, w_1 \models B_b q$, $\quad \mathcal{M}, w_1 \models B_c p$.

$C_{\{b,c\}}(w_1) = \{w_1\}$

$\mathcal{M}, s \vDash D^{\forall}_{G}\varphi$ iff $\forall G' \subseteq^{max}_{s} G, \forall s' \in C_{G'}(s)$: $\mathcal{M}, s' \vDash \varphi$.

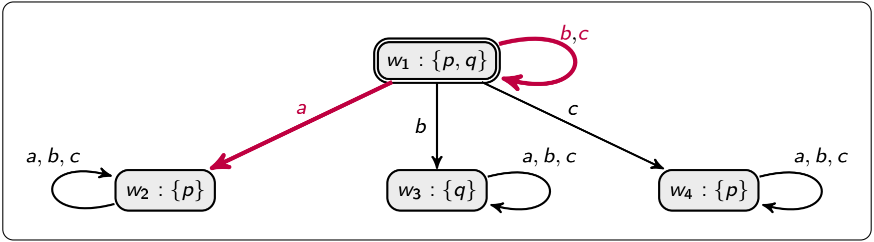


$G = \{a, b, c\}$:
$C_a(w_1) = \{w_2\}$, $C_b(w_1) = \{w_1, w_3\}$, $C_c(w_1) = \{w_1, w_4\}$

- $\mathcal{M}, w_1 \vDash B_a p \wedge B_a \neg q$, $\mathcal{M}, w_1 \vDash B_b q$, $\mathcal{M}, w_1 \vDash B_c p$.

$C_{\{b,c\}}(w_1) = \{w_1\}$
$C_a(w_1) \cup C_{\{b,c\}}(w_1) = \{w_1, w_2\}$

$\mathcal{M}, s \models D^{\forall}_G \varphi$ iff $\forall G' \subseteq^{max}_s G$, $\forall s' \in C_{G'}(s)$: $\mathcal{M}, s' \models \varphi$.



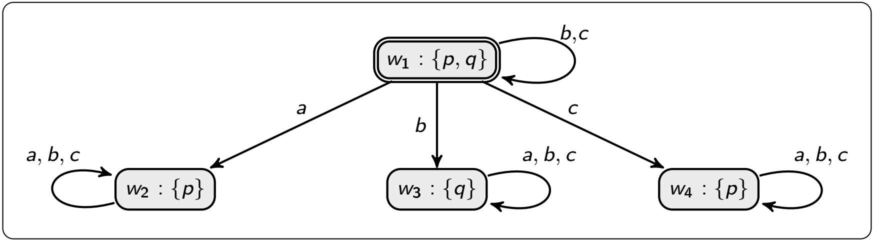
$G = \{a, b, c\}$:
$C_a(w_1) = \{w_2\}$, $C_b(w_1) = \{w_1, w_3\}$, $C_c(w_1) = \{w_1, w_4\}$

- $\mathcal{M}, w_1 \models B_a p \wedge B_a \neg q$, $\quad \mathcal{M}, w_1 \models B_b q$, $\quad \mathcal{M}, w_1 \models B_c p$.

  $C_{\{b,c\}}(w_1) = \{w_1\}$
  $C_a(w_1) \cup C_{\{b,c\}}(w_1) = \{w_1, w_2\}$

- $\mathcal{M}, w_1 \models D^{\forall}_G\, p$, $\quad \mathcal{M}, w_1 \models \neg D^{\forall}_G\, q$

$\mathcal{M}, s \models D^{\forall}_{G}\varphi$ iff $\forall G' \subseteq^{max}_{s} G, \forall s' \in C_{G'}(s)$: $\mathcal{M}, s' \models \varphi$.



$G = \{a, b, c\}$:
$C_a(w_1) = \{w_2\}$, $C_b(w_1) = \{w_1, w_3\}$, $C_c(w_1) = \{w_1, w_4\}$

- $\mathcal{M}, w_1 \models B_a p \wedge B_a \neg q$, $\mathcal{M}, w_1 \models B_b q$, $\mathcal{M}, w_1 \models B_c p$.

  $C_{\{b,c\}}(w_1) = \{w_1\}$
  $C_a(w_1) \cup C_{\{b,c\}}(w_1) = \{w_1, w_2\}$

- $\mathcal{M}, w_1 \models D^{\forall}_{G}\, p$, $\mathcal{M}, w_1 \models \neg\, D^{\forall}_{G}\, q$
- $\mathcal{M}, w_1 \models D_G\, p \wedge D_G\, q$, but also $\mathcal{M}, w_1 \models D_G \perp$.

Some results

- Consistent if some agent is consistent:

$$\vDash D_G^{\forall} \bot \leftrightarrow \bigwedge_{a \in G} B_a \bot$$

- Equivalent to standard distributed belief on the class of reflexive models ($\mathbf{T}$):

$$\mathbf{T} \vDash D_G^{\forall} \varphi \leftrightarrow D_G \varphi.$$

- Inheriting relational properties from individual relations (section 3).
- Standard distributed belief is strictly more expressive than cautious distributed belief (section 4).

Cautious and standard distributed belief

- $\mathcal{L}_D$ vs $\mathcal{L}_{D^\forall}$.
- Cautious distributed belief definable using standard distributed belief:

$$\models D^\forall_G \varphi \leftrightarrow \bigwedge_{G' \subseteq G} \Big((\neg D_{G'} \bot \wedge \bigwedge_{G' \subset H \subseteq G} D_H \bot) \rightarrow D_{G'} \varphi \Big).$$

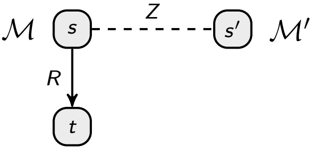
  Thus, $\mathcal{L}_{D^\forall} \preccurlyeq \mathcal{L}_D$.
- $\mathcal{L}_D \preccurlyeq \mathcal{L}_{D^\forall}$?
- Bisimulation to capture modal equivalence for $\mathcal{L}_{D^\forall}$.

Cautious collective bisimulation

- Bisimulation: standard notion to semantically capture modal equivalence.
- Relation Z between (points in) models, satisfying conditions **Atom**, **Forth** and **Back**.

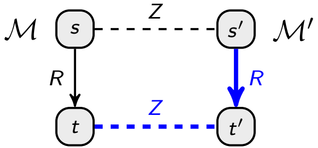
Cautious collective bisimulation

- Bisimulation: standard notion to semantically capture modal equivalence.
- Relation Z between (points in) models, satisfying conditions **Atom**, **Forth** and **Back**.



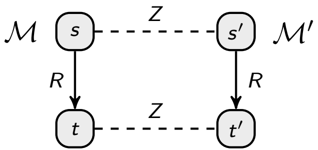
Cautious collective bisimulation

- Bisimulation: standard notion to semantically capture modal equivalence.
- Relation Z between (points in) models, satisfying conditions **Atom**, **Forth** and **Back**.



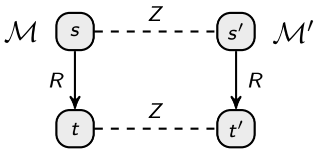
Cautious collective bisimulation

- Bisimulation: standard notion to semantically capture modal equivalence.
- Relation Z between (points in) models, satisfying conditions **Atom**, **Forth** and **Back**.



Cautious collective bisimulation

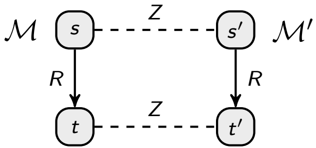
- Bisimulation: standard notion to semantically capture modal equivalence.
- Relation Z between (points in) models, satisfying conditions **Atom**, **Forth** and **Back**.



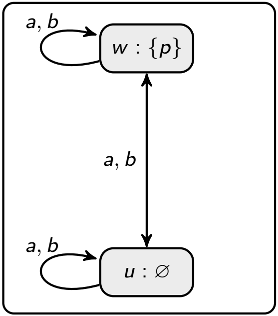
- Collective cautious bisimilarity:
  Forth If Zss', then $\forall G \subseteq A$, $\forall t \in \mathrm{D}(\mathcal{M})$:
  $sR^{\forall}_{G}t \Rightarrow \exists t'$: $s'R^{\forall}_{G}t'$ and Ztt'.

Cautious collective bisimulation

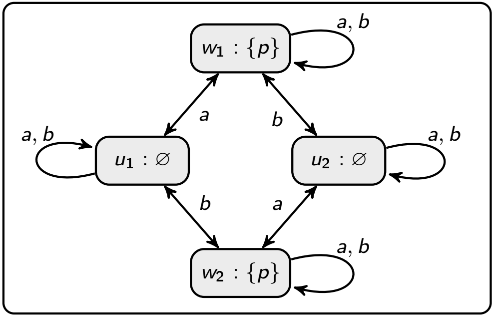
- Bisimulation: standard notion to semantically capture modal equivalence.
- Relation Z between (points in) models, satisfying conditions **Atom**, **Forth** and **Back**.



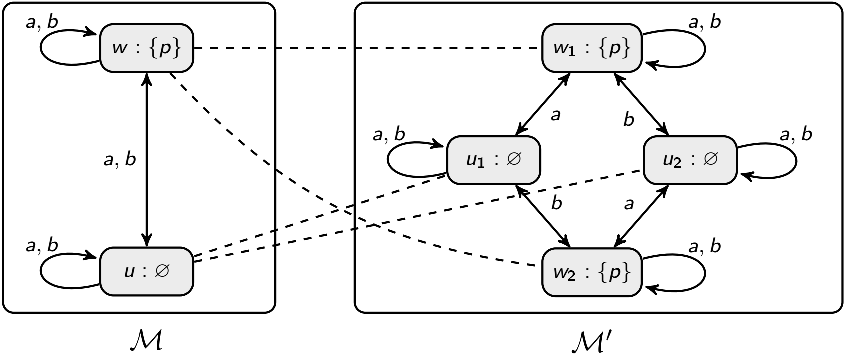
- Collective cautious bisimilarity:
  Forth If Zss', then $\forall G \subseteq A$, $\forall t \in \mathrm{D}(\mathcal{M})$:
  ~~$sR^{\forall}_{G}t \Rightarrow \exists t'$: $s'R^{\forall}_{G}t'$ and Ztt'.~~
  $\exists H \subseteq^{max}_{s} G : t \in C_H(s) \Rightarrow \exists H' \subseteq^{max}_{s'} G$, $\exists t' \in C_{H'}(s')$ and Ztt'.

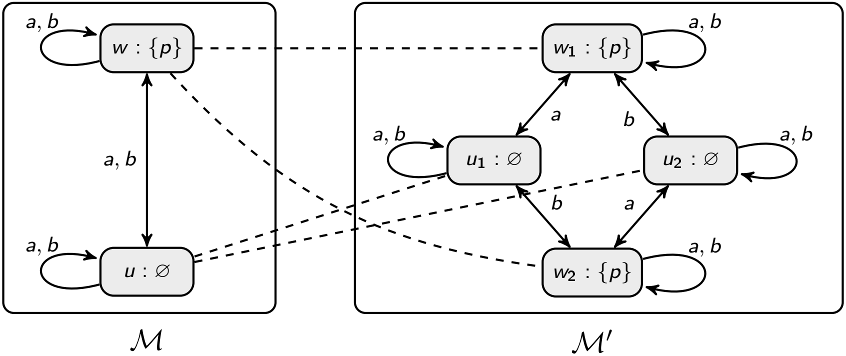


$\mathcal{M}$

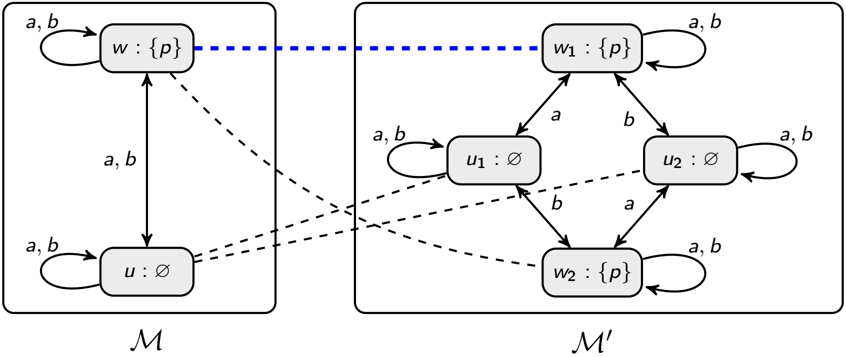


$\mathcal{M}'$

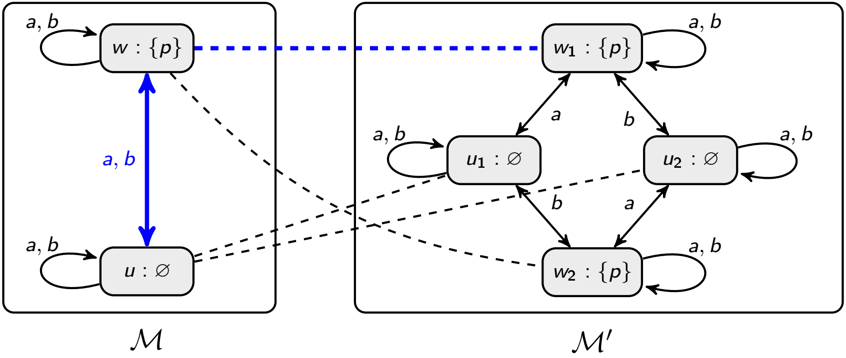

a, b
w : {p}
a, b
a, b
u : ∅
M
w1 : {p}
a, b
a
b
a, b
u1 : ∅
u2 : ∅
a, b
b
a
w2 : {p}
a, b
M′



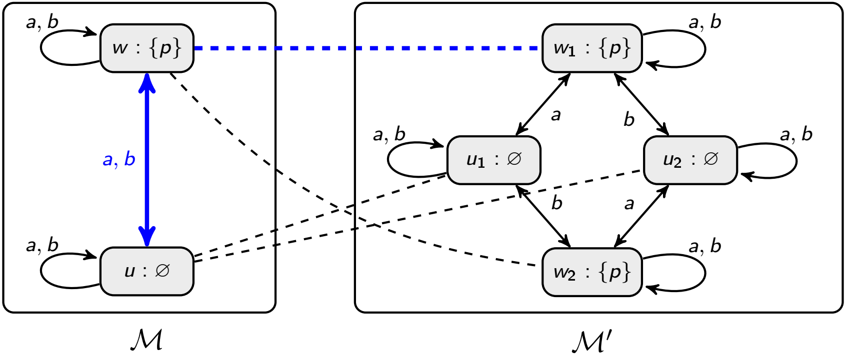
- **Forth**. If Zss', then $\forall G \subseteq A$, $\forall t \in \mathsf{D}(\mathcal{M})$: $\exists H \subseteq^{max}_{s} G$ such that $t \in C_H(s) \Rightarrow Ztt'$ and $\exists H' \subseteq^{max}_{s'} G$, $\exists t' \in C_{H'}(s')$.



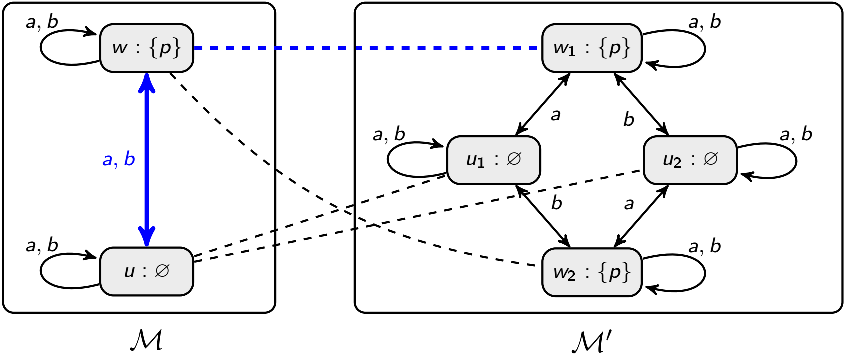
- **Forth**. If Zss', then $\forall G \subseteq A, \forall t \in \mathrm{D}(\mathcal{M})$: $\exists H \subseteq_s^{max} G$ such that $t \in C_H(s) \Rightarrow Ztt'$ and $\exists H' \subseteq_{s'}^{max} G$, $\exists t' \in C_{H'}(s')$.



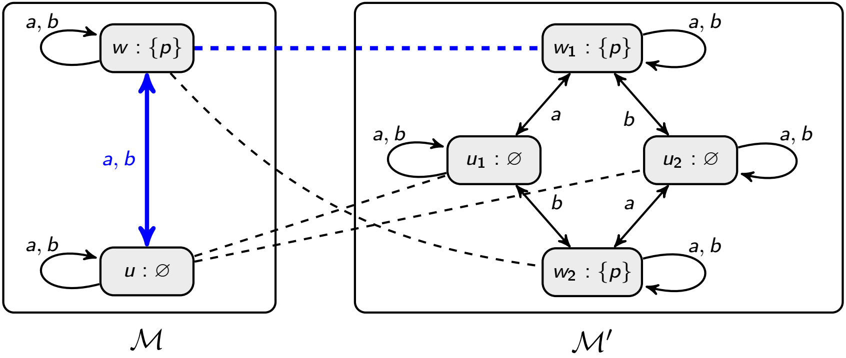
- **Forth**. If Zss', then $\forall G \subseteq A$, $\forall t \in \mathrm{D}(\mathcal{M})$: $\exists H \subseteq_s^{max} G$ such that $t \in C_H(s) \Rightarrow Ztt'$ and $\exists H' \subseteq_{s'}^{max} G$, $\exists t' \in C_{H'}(s')$.
- $\{a, b\} \subseteq_w^{max} \{a, b\}$ and $u \in C_{\{a,b\}}(w)$.



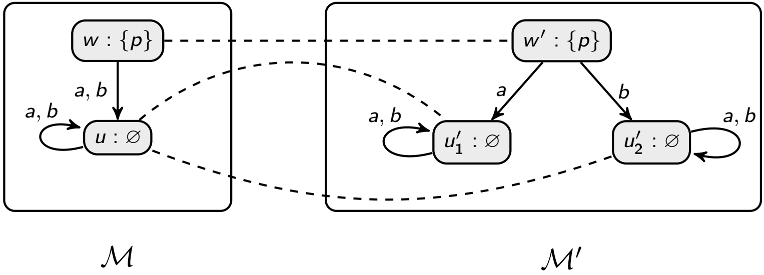
- **Forth**. If Zss', then $\forall G \subseteq A$, $\forall t \in \mathrm{D}(\mathcal{M})$: $\exists H \subseteq_s^{max} G$ such that $t \in C_H(s) \Rightarrow Ztt'$ and $\exists H' \subseteq_{s'}^{max} G$, $\exists t' \in C_{H'}(s')$.
- $\{a, b\} \subseteq_w^{max} \{a, b\}$ and $u \in C_{\{a,b\}}(w)$.
- However, $\{a, b\} \subseteq_{w_1}^{max} \{a, b\}$ (alone), and $C_{\{a,b\}}(w_1) = \{w_1\}$.

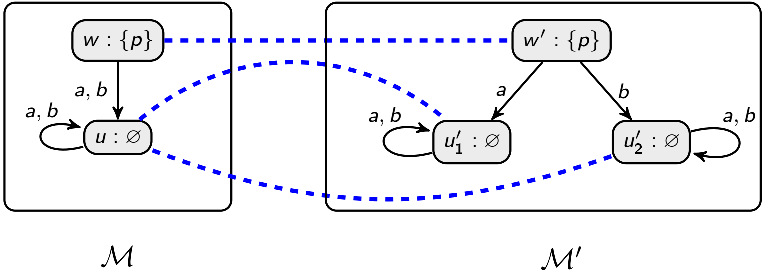


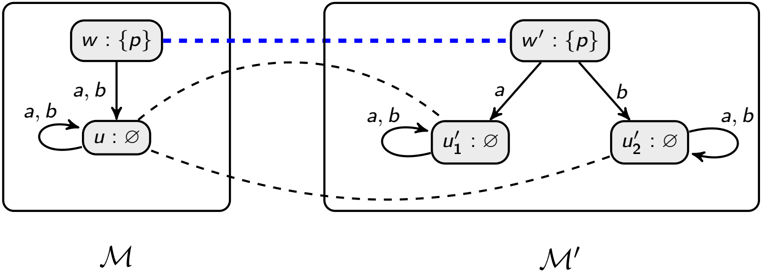
- **Forth**. If Zss', then $\forall G \subseteq A$, $\forall t \in \mathrm{D}(\mathcal{M})$: $\exists H \subseteq_s^{max} G$ such that $t \in C_H(s) \Rightarrow Ztt'$ and $\exists H' \subseteq_{s'}^{max} G$, $\exists t' \in C_{H'}(s')$.
- $\{a, b\} \subseteq_w^{max} \{a, b\}$ and $u \in C_{\{a,b\}}(w)$.
- However, $\{a, b\} \subseteq_{w_1}^{max} \{a, b\}$ (alone), and $C_{\{a,b\}}(w_1) = \{w_1\}$.
- Zuw_1 fails by **Atom**.



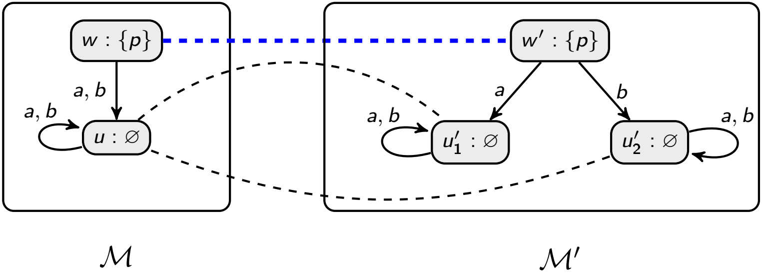
- **Forth**. If Zss', then $\forall G \subseteq A$, $\forall t \in \mathrm{D}(\mathcal{M})$: $\exists H \subseteq_s^{max} G$ such that $t \in C_H(s) \Rightarrow Ztt'$ and $\exists H' \subseteq_{s'}^{max} G$, $\exists t' \in C_{H'}(s')$.
- $\{a, b\} \subseteq_w^{max} \{a, b\}$ and $u \in C_{\{a,b\}}(w)$.
- However, $\{a, b\} \subseteq_{w_1}^{max} \{a, b\}$ (alone), and $C_{\{a,b\}}(w_1) = \{w_1\}$.
- Zuw_1 fails by **Atom**.


$w : \{p\}$
a, b
a, b
$u : \varnothing$
$w' : \{p\}$
a
b
a, b
$u'_1 : \varnothing$
$u'_2 : \varnothing$
a, b
$\mathcal{M}$
$\mathcal{M}'$

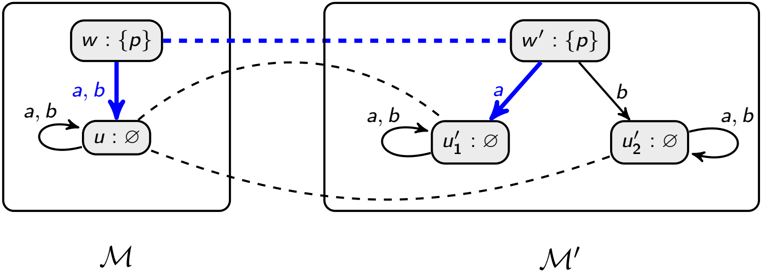

w : {p}
a, b
a, b
u : ∅
w′ : {p}
a
b
a, b
u′1 : ∅
u′2 : ∅
a, b
𝓜
𝓜′



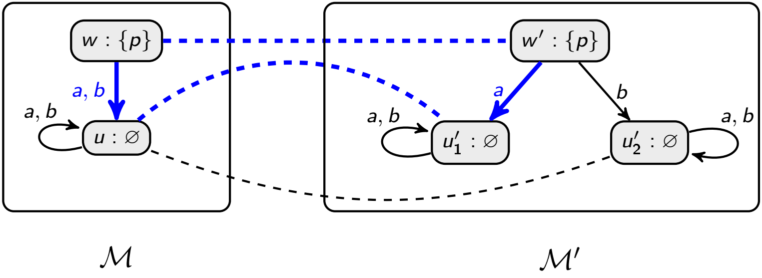
- (w, w').



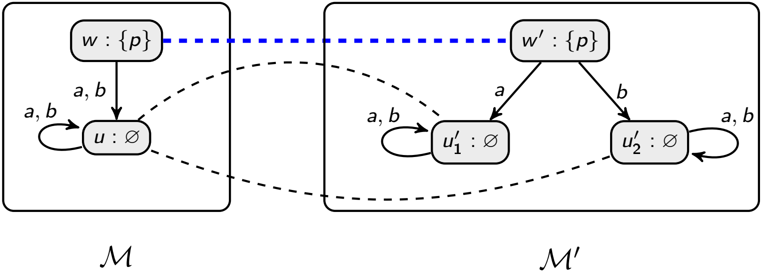
- (w, w').
- $G = \{a\}$: $C_a(w) = \{u\}$, $C_a(w') = \{u'_1\}$ and Zuu'_1.



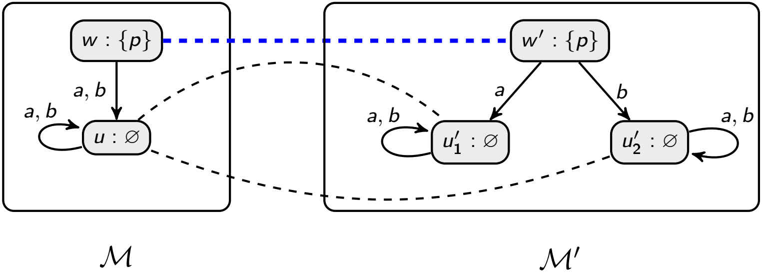
- (w, w').
- $G = \{a\}$: $C_a(w) = \{u\}$, $C_a(w') = \{u'_1\}$ and Zuu'_1.



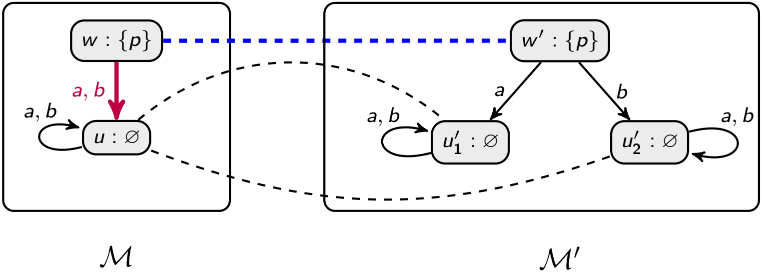
- (w, w').
- $G = \{a\}$: $C_a(w) = \{u\}$, $C_a(w') = \{u'_1\}$ and Zuu'_1.



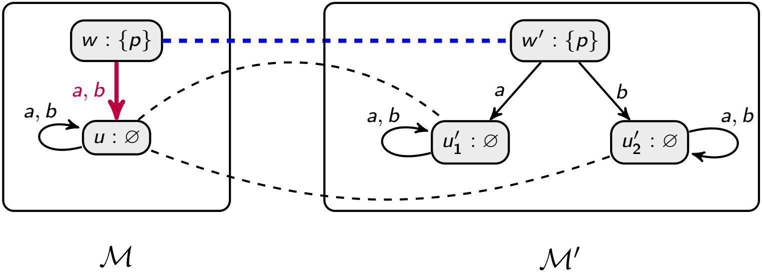
- (w, w').
- $G = \{a\}$: $C_a(w) = \{u\}$, $C_a(w') = \{u'_1\}$ and Zuu'_1.



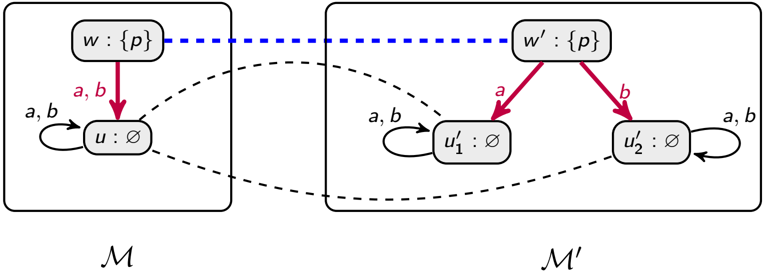
- (w, w').
- $G = \{a\}$: $C_a(w) = \{u\}$, $C_a(w') = \{u'_1\}$ and Zuu'_1.
- $G = \{a, b\}$: $G \subseteq^{max}_{w} G$, $C_G(w) = \{u\}$. $\{a\} \subseteq^{max}_{w'} G$ and $\{b\} \subseteq^{max}_{w} G$. $C_a(w') \cup C_b(w') = \{u'_1, u'_2\}$. Zuu'_1 and Zuu'_2.



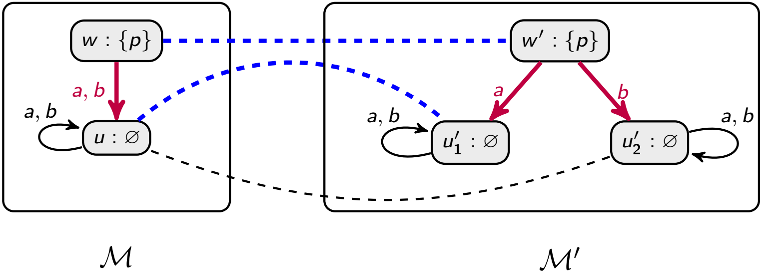
- (w, w').
- $G = \{a\}$: $C_a(w) = \{u\}$, $C_a(w') = \{u'_1\}$ and Zuu'_1.
- $G = \{a, b\}$: $G \subseteq^{max}_{w} G$, $C_G(w) = \{u\}$. $\{a\} \subseteq^{max}_{w'} G$ and $\{b\} \subseteq^{max}_{w} G$. $C_a(w') \cup C_b(w') = \{u'_1, u'_2\}$. Zuu'_1 and Zuu'_2.



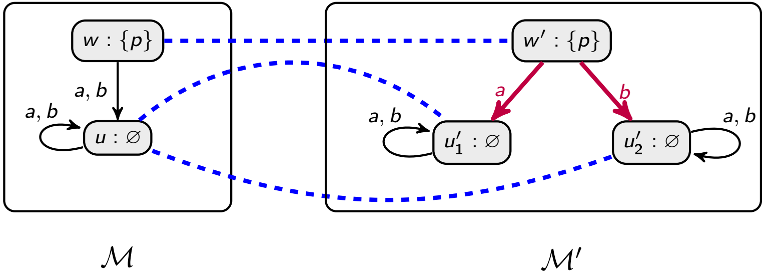
- (w, w').
- $G = \{a\}$: $C_a(w) = \{u\}$, $C_a(w') = \{u'_1\}$ and Zuu'_1.
- $G = \{a, b\}$: $G \subseteq^{max}_{w} G$, $C_G(w) = \{u\}$. $\{a\} \subseteq^{max}_{w'} G$ and $\{b\} \subseteq^{max}_{w} G$. $C_a(w') \cup C_b(w') = \{u'_1, u'_2\}$.
  Zuu'_1 and Zuu'_2.



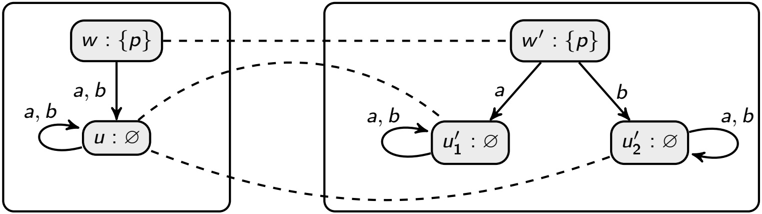
- (w, w').
- $G = \{a\}$: $C_a(w) = \{u\}$, $C_a(w') = \{u'_1\}$ and Zuu'_1.
- $G = \{a, b\}$: $G \subseteq^{max}_{w} G$, $C_G(w) = \{u\}$. $\{a\} \subseteq^{max}_{w'} G$ and $\{b\} \subseteq^{max}_{w} G$. $C_a(w') \cup C_b(w') = \{u'_1, u'_2\}$.
  Zuu'_1 and Zuu'_2.



- (w, w').
- $G = \{a\}$: $C_a(w) = \{u\}$, $C_a(w') = \{u'_1\}$ and Zuu'_1.
- $G = \{a, b\}$: $G \subseteq^{max}_w G$, $C_G(w) = \{u\}$. $\{a\} \subseteq^{max}_{w'} G$ and $\{b\} \subseteq^{max}_w G$. $C_a(w') \cup C_b(w') = \{u'_1, u'_2\}$. Zuu'_1 and Zuu'_2.



- (w, w').
- $G = \{a\}$: $C_a(w) = \{u\}$, $C_a(w') = \{u'_1\}$ and Zuu'_1.
- $G = \{a, b\}$: $G \subseteq^{max}_{w} G$, $C_G(w) = \{u\}$. $\{a\} \subseteq^{max}_{w'} G$ and $\{b\} \subseteq^{max}_{w} G$. $C_a(w') \cup C_b(w') = \{u'_1, u'_2\}$.
  Zuu'_1 and Zuu'_2.



$\mathcal{M}$ $\mathcal{M}'$

- (w, w').
- $G = \{a\}$: $C_a(w) = \{u\}$, $C_a(w') = \{u'_1\}$ and Zuu'_1.
- $G = \{a, b\}$: $G \subseteq^{max}_{w} G$, $C_G(w) = \{u\}$. $\{a\} \subseteq^{max}_{w'} G$ and $\{b\} \subseteq^{max}_{w} G$. $C_a(w') \cup C_b(w') = \{u'_1, u'_2\}$. Zuu'_1 and Zuu'_2.
- Note: $\mathcal{M}, w \not\models D_{\{a,b\}} \bot$ and $\mathcal{M}', w' \models D_{\{a,b\}} \bot$.

The difference in expressivity

Corollary 2 *$\mathcal{L}_D$ is strictly more expressive than $\mathcal{L}_{D^\forall}$ (symbols: $\mathcal{L}_{D^\forall} \prec \mathcal{L}_D$).* ■

- Adding an inconsistency constant:

$$\mathcal{M}, s \vDash \asymp_G \quad \text{iff} \quad C_G(s) = \varnothing.$$

- Standard distributed belief is definable in terms of $\mathcal{L}_{D^\forall,\asymp}$:

$$\vDash D_G\,\varphi \leftrightarrow (\asymp_G \vee D^\forall_G\,\varphi).$$

Proposition 7 *$\mathcal{L}_{D^\forall,\asymp}$ and $\mathcal{L}_D$ are equally expressive (symbols: $\mathcal{L}_{D^\forall,\asymp} \approx \mathcal{L}_D$).* ■

Summing up

- Introducing cautious distributed belief.
- Comparing with individual and standard distributed belief.
- Relational properties.
- Expressivity and bisimulation.

Conclusion

Future work:

- Axiomatising $\mathcal{L}_{D^{\forall}}$?
- “Bold” distributed belief.
- Resolving cautious distributed belief.