A Labelled Sequent Calculus for Public Announcement Logic

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Basics of epistemic logic(EL) and public announcement logic(PAL)

Definition (Language of EL and PAL)

Let Prop be a denumerable set of variables and Ag a finite set of agents. Language $\mathcal{L}_{\text{EL}}$ for epistemic logic is defined inductively as follows:

$$\mathcal{L} ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \rightarrow \varphi \mid K_a\varphi$$

Language $\mathcal{L}_{\text{PAL}}$ for public announcement logic is EL plus the public announcement formulas $[\varphi]\varphi$.

Epistemic model

$$\mathcal{M} = \{W, \{\sim_a\}_{a \in \text{Ag}}, V\}$$

Axiomatization of EL and PAL

EL is axiomatized by (Tau), (K), (4), (T), (5), (MP) and (GK_a). PAL is axiomatized by the axiomatization for EL plus reduction axioms (R1–6):

(Tau)	Classical propositional tautologies.
(K)	$K_a(\varphi \to \psi) \to (K_a\varphi \to K_a\psi)$
(4)	$K_a\varphi \to K_aK_a\varphi$
(T)	$K_a\varphi \to \varphi$
(5)	$\neg K_a\varphi \to K_a\neg K_a\varphi$
(MP)	From φ and $\varphi \to \psi$ infer ψ.
(GK_a)	From φ infer $K_a\varphi$.

(R1)	$[\varphi]p \leftrightarrow (\varphi \to p)$
(R2)	$[\varphi]\neg\psi \leftrightarrow (\varphi \to \neg[\varphi]\psi)$
(R3)	$[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$
(R4)	$[\varphi](\psi \to \chi) \leftrightarrow ([\varphi]\psi \to [\varphi]\chi)$
(R5)	$[\varphi]K_a\psi \leftrightarrow (\varphi \to K_a[\varphi]\psi)$
(R6)	$[\varphi][\psi]\chi \leftrightarrow [\varphi \wedge [\varphi]\psi]\chi$

Labelled Sequent Calculus

A labelled sequent calculus is based on the internalization of Kripke semantics.

Definition

A relational atom is of the form $x \sim_a y$. A labelled formula is of the form $x : \varphi$.

- Interpretation $\tau_{\mathcal{M}}$ of labelled formulas on an epistemic model $\mathcal{M}$
  - $\tau_{\mathcal{M}}(x : \varphi) = \mathcal{M}, x \models \varphi$
  - $\tau_{\mathcal{M}}(x \sim_a y) = x \sim_a y$
- A labelled sequent $\sigma_1, \ldots, \sigma_m \Rightarrow \delta_1, \ldots, \delta_n$ is *valid* if the following is true:

$$\forall \mathcal{M} \forall x_1 \ldots \forall x_k [\tau_{\mathcal{M}}(\sigma_1) \wedge \ldots \wedge \tau_{\mathcal{M}}(\sigma_m) \rightarrow \tau_{\mathcal{M}}(\delta_1) \vee \ldots \vee \tau_{\mathcal{M}}(\delta_n)]$$

Labelled Sequent Calculus G_{EL} for **EL**

- Initial sequents:

$$x : p, \Gamma \Rightarrow \Delta, x : p \qquad\qquad x \sim_a y, \Gamma \Rightarrow \Delta, x \sim_a y$$

- Propositional rules:

$$(\neg l)\ \frac{\Gamma \Rightarrow \Delta, x : \varphi}{x : \neg\varphi, \Gamma \Rightarrow \Delta} \qquad (\neg r)\ \frac{x : \varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, x : \neg\varphi}$$

$$(\wedge l)\ \frac{x : \varphi_1, x : \varphi_2, \Gamma \Rightarrow \Delta}{x : \varphi_1 \wedge \varphi_2, \Gamma \Rightarrow \Delta} \qquad (\wedge r)\ \frac{\Gamma \Rightarrow \Delta, x : \varphi_1 \quad \Gamma \Rightarrow \Delta, x : \varphi_2}{\Gamma \Rightarrow \Delta, x : \varphi_1 \wedge \varphi_2}$$

$$(\to l)\ \frac{\Gamma \Rightarrow \Delta, x : \varphi \quad x : \psi, \Gamma \Rightarrow \Delta}{x : \varphi \to \psi, \Gamma \Rightarrow \Delta} \qquad (\to r)\ \frac{x : \varphi, \Gamma \Rightarrow \Delta, x : \psi}{\Gamma \Rightarrow \Delta, x : \varphi \to \psi}$$

Labelled Sequent Calculus G_{EL} for **EL**

- Modal rules:

$$(K_a \Rightarrow) \ \frac{y:\varphi, x:K_a\varphi, x \sim_a y, \Gamma \Rightarrow \Delta}{x:K_a\varphi, x \sim_a y, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow K_a)^{\dagger} \ \frac{x \sim_a y, \Gamma \Rightarrow \Delta, y:\varphi}{\Gamma \Rightarrow \Delta, x:K_a\varphi}$$

† y does not occur in the conclusion of $(\Rightarrow K_a)$

- Relational rules:

$$(\mathrm{Ref}_a) \ \frac{x \sim_a x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

$$(\mathrm{Trans}_a) \ \frac{x \sim_a z, x \sim_a y, y \sim_a z, \Gamma \Rightarrow \Delta}{x \sim_a y, y \sim_a z, \Gamma \Rightarrow \Delta}$$

$$(\mathrm{Sym}_a) \ \frac{y \sim_a x, x \sim_a y, \Gamma \Rightarrow \Delta}{x \sim_a y, \Gamma \Rightarrow \Delta}$$

Labelled Sequent Calculus G_{PAL} for **PAL**

- **Basic idea**: construct sequent rules for reduction axioms
- G_{PAL} is G_{EL} plus 6 pairs of rules, each pair corresponding a reduction axiom:

$$(R1)[\varphi]p \leftrightarrow (\varphi \rightarrow p)$$

$$\frac{x:\varphi \rightarrow p, \Gamma \Rightarrow \Delta}{x:[\varphi]p, \Gamma \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \Delta, x:\varphi \rightarrow p}{\Gamma \Rightarrow \Delta, x:[\varphi]p}$$

$$(R1l)\ \frac{\Gamma \Rightarrow \Delta, x:\varphi \quad x:p, \Gamma \Rightarrow \Delta}{x:[\varphi]p, \Gamma \Rightarrow \Delta} \quad (R1r)\ \frac{x:\varphi, \Gamma \Rightarrow \Delta, x:p}{\Gamma \Rightarrow \Delta, x:[\varphi]p}$$

Labelled Sequent Calculus G_{PAL} for **PAL**

- G_{EL} plus the following reduction rules

$$(R1l)\ \frac{\Gamma \Rightarrow \Delta, x:\varphi \quad x:p, \Gamma \Rightarrow \Delta}{x:[\varphi]p, \Gamma \Rightarrow \Delta} \qquad (R1r)\ \frac{x:\varphi, \Gamma \Rightarrow \Delta, x:p}{\Gamma \Rightarrow \Delta, x:[\varphi]p}$$

$$(R2l)\ \frac{\Gamma \Rightarrow \Delta, x:\varphi \quad x:\neg[\varphi]\psi, \Gamma \Rightarrow \Delta}{x:[\varphi]\neg\psi, \Gamma \Rightarrow \Delta} \qquad (R2r)\ \frac{x:\varphi, \Gamma \Rightarrow \Delta, x:\neg[\varphi]\psi}{\Gamma \Rightarrow \Delta, x:[\varphi]\neg\psi}$$

$$(R3l)\ \frac{x:[\varphi]\psi_1, x:[\varphi]\psi_2, \Gamma \Rightarrow \Delta}{x:[\varphi](\psi_1 \wedge \psi_2), \Gamma \Rightarrow \Delta} \qquad (R3r)\ \frac{\Gamma \Rightarrow \Delta, x:[\varphi]\psi_1 \quad \Gamma \Rightarrow \Delta, x:[\varphi]\psi_2}{\Gamma \Rightarrow \Delta, x:[\varphi](\psi_1 \wedge \psi_2)}$$

$$(R4l)\ \frac{\Gamma \Rightarrow \Delta, x:[\varphi]\psi_1 \quad x:[\varphi]\psi_2, \Gamma \Rightarrow \Delta}{x:[\varphi](\psi_1 \rightarrow \psi_2), \Gamma \Rightarrow \Delta} \qquad (R4r)\ \frac{x:[\varphi]\psi_1, \Gamma \Rightarrow \Delta, x:[\varphi]\psi_2}{\Gamma \Rightarrow \Delta, [\varphi](\psi_1 \rightarrow \psi_2)}$$

$$(R5l)\ \frac{\Gamma \Rightarrow \Delta, x:\varphi \quad K_a[\varphi]\psi, \Gamma \Rightarrow \Delta}{x:[\varphi]K_a\psi, \Gamma \Rightarrow \Delta} \qquad (R5r)\ \frac{x:\varphi, \Gamma \Rightarrow \Delta, x:K_a[\varphi]\psi}{\Gamma \Rightarrow \Delta, x:[\varphi]K_a\psi}$$

$$(R6l)\ \frac{x:[\varphi \wedge [\varphi]\psi]\chi, \Gamma \Rightarrow \Delta}{x:[\varphi][\psi]\chi, \Gamma \Rightarrow \Delta} \qquad (R6r)\ \frac{\Gamma \Rightarrow \Delta, x:[\varphi \wedge [\varphi]\psi]\chi}{\Gamma \Rightarrow \Delta, x:[\varphi][\psi]\chi}$$

Labelled Sequent Calculus G_{PAL} for **PAL**

- G_{EL} plus the following reduction rules

$$(R1l)\ \frac{\Gamma \Rightarrow \Delta, x:\varphi \quad x:p,\Gamma \Rightarrow \Delta}{x:[\varphi]p,\Gamma \Rightarrow \Delta} \qquad (R1r)\ \frac{x:\varphi,\Gamma \Rightarrow \Delta, x:p}{\Gamma \Rightarrow \Delta, x:[\varphi]p}$$

$$(R2l)\ \frac{\Gamma \Rightarrow \Delta, x:\varphi \quad x:\neg[\varphi]\psi,\Gamma \Rightarrow \Delta}{x:[\varphi]\neg\psi,\Gamma \Rightarrow \Delta} \qquad (R2r)\ \frac{x:\varphi,\Gamma \Rightarrow \Delta, x:\neg[\varphi]\psi}{\Gamma \Rightarrow \Delta, x:[\varphi]\neg\psi}$$

$$(R3l)\ \frac{x:[\varphi]\psi_1, x:[\varphi]\psi_2,\Gamma \Rightarrow \Delta}{x:[\varphi](\psi_1 \wedge \psi_2),\Gamma \Rightarrow \Delta} \qquad (R3r)\ \frac{\Gamma \Rightarrow \Delta, x:[\varphi]\psi_1 \quad \Gamma \Rightarrow \Delta, x:[\varphi]\psi_2}{\Gamma \Rightarrow \Delta, x:[\varphi](\psi_1 \wedge \psi_2)}$$

$$(R4l)\ \frac{\Gamma \Rightarrow \Delta, x:[\varphi]\psi_1 \quad x:[\varphi]\psi_2,\Gamma \Rightarrow \Delta}{x:[\varphi](\psi_1 \rightarrow \psi_2),\Gamma \Rightarrow \Delta} \qquad (R4r)\ \frac{x:[\varphi]\psi_1,\Gamma \Rightarrow \Delta, x:[\varphi]\psi_2}{\Gamma \Rightarrow \Delta, [\varphi](\psi_1 \rightarrow \psi_2)}$$

$$(R5l)\ \frac{\Gamma \Rightarrow \Delta, x:\varphi \quad x:K_a[\varphi]\psi,\Gamma \Rightarrow \Delta}{x:[\varphi]K_a\psi,\Gamma \Rightarrow \Delta} \qquad (R5r)\ \frac{x:\varphi,\Gamma \Rightarrow \Delta, x:K_a[\varphi]\psi}{\Gamma \Rightarrow \Delta, x:[\varphi]K_a\psi}$$

$$(R6l)\ \frac{x:[\varphi \wedge [\varphi]\psi]\chi,\Gamma \Rightarrow \Delta}{x:[\varphi][\psi]\chi,\Gamma \Rightarrow \Delta} \qquad (R6r)\ \frac{\Gamma \Rightarrow \Delta, x:[\varphi \wedge [\varphi]\psi]\chi}{\Gamma \Rightarrow \Delta, x:[\varphi][\psi]\chi}$$

Labelled Sequent Calculus G_{PAL} for **PAL**

Definition (Complexity)

Let φ be an $\mathcal{L}_{\text{PAL}}$ formula, the *complexity* $c(\varphi)$ of φ is defined as follows:

$$c(p) = 1 \quad c(\varphi \to \psi) = 1 + \max\{c(\varphi), c(\psi)\}$$

$$c(\neg\varphi) = 1 + c(\varphi) \quad c(\varphi \wedge \psi) = 1 + \max\{c(\varphi), c(\psi)\}$$

$$\underline{c(K_a\varphi) = 1 + c(\varphi)} \quad \underline{c([\varphi]\psi) = (4 + c(\varphi)) \cdot c(\psi)}$$

Labelled Sequent Calculus G_{PAL} for **PAL**

Example (Proof of (R5:$[\varphi]K_a\psi \leftrightarrow \varphi \rightarrow K_a[\varphi]\psi$))

left-to-right direction:

$$
\dfrac{
  \dfrac{
    x \sim_a y, x : \varphi \Rightarrow y : [\varphi]\psi, x : \varphi
    \qquad
    \dfrac{y : [\varphi]\psi, x : K_a[\varphi]\psi, x \sim_a y, x : \varphi \Rightarrow y : [\varphi]\psi}{x : K_a[\varphi]\psi, x \sim_a y, x : \varphi \Rightarrow y : [\varphi]\psi}\,(K_a \Rightarrow)
  }{
    \dfrac{x \sim_a y, x : [\varphi]K_a\psi, x : \varphi \Rightarrow y : [\varphi]\psi}{
      \dfrac{x : [\varphi]K_a\psi, x : \varphi \Rightarrow x : K_a[\varphi]\psi}{x : [\varphi]K_a\psi \Rightarrow x : \varphi \rightarrow K_a[\varphi]\psi}\,(\Rightarrow\rightarrow)
    }\,(\Rightarrow K_a)
  }\,(R5 \Rightarrow)
}{
  \Rightarrow x : [\varphi]K_a\psi \rightarrow (\varphi \rightarrow K_a[\varphi]\psi)
}\,(\Rightarrow\rightarrow)
$$

Labelled Sequent Calculus G_{PAL} for **PAL**

Example (Proof of (R5:$[\varphi]K_a\psi \leftrightarrow \varphi \to K_a[\varphi]\psi$))

Right-to-left direction

$$\dfrac{\dfrac{\dfrac{x : \varphi \Rightarrow x : K_a[\varphi]\psi, x : \varphi \qquad x : K_a[\varphi]\psi, x : \varphi \Rightarrow x : K_a[\varphi]\psi}{x : \varphi, x : \varphi \to K_a[\varphi]\psi \Rightarrow x : K_a[\varphi]\psi}\,(\to\Rightarrow)}{x : \varphi \to K_a[\varphi]\psi \Rightarrow x : [\varphi]K_a\psi}\,(\Rightarrow R5)}{\Rightarrow x : (\varphi \to K_a[\varphi]\psi) \to [\varphi]K_a\psi}\,(\Rightarrow\to)$$

Labelled Sequent Calculus G_{PAL} for **PAL**

In light of the reduction axioms, we define a translation t from $\mathcal{L}_{PAL}$−formulas to $\mathcal{L}_{EL}$−formulas:

Definition

$$
\begin{array}{ll}
t(p) = p & t([\varphi]p) = t(\varphi \to p) \\
t(\neg\varphi) = \neg t(\varphi) & t([\varphi]\neg\psi) = t(\varphi \to \neg[\varphi]\psi) \\
t(\varphi \wedge \psi) = t(\varphi) \wedge t(\psi) & t([\varphi](\psi \wedge \chi)) = t([\varphi]\psi \wedge [\varphi]\chi) \\
t(\varphi \to \psi) = t(\varphi) \to t(\psi) & t([\varphi](\psi \to \chi)) = t([\varphi]\psi \to [\varphi]\chi) \\
t(K_a\varphi) = K_a t(\varphi) & t([\varphi]K_a\psi) = t(\varphi \to K_a[\varphi]\psi) \\
 & t([\varphi][\psi]\chi) = t([\varphi \wedge [\varphi]\psi]\chi)
\end{array}
$$

- Translation t can be extended to relational atoms and labelled formulas:
  - $t(x \sim_a y) = x \sim_a y$
  - $t(x : \varphi) = x : t(\varphi)$

Labelled Sequent Calculus G_{PAL} for **PAL**

Theorem

For any $\mathcal{L}_{\mathsf{PAL}}$-sequent $\Gamma \Rightarrow \Delta$,

1. *if $G_{\mathsf{EL}} \vdash t(\Gamma) \Rightarrow t(\Delta)$, then $G_{\mathsf{PAL}} \vdash \Gamma \Rightarrow \Delta$;*
2. *if $G_{\mathsf{PAL}} \vdash_h \Gamma \Rightarrow \Delta$, then $G_{\mathsf{EL}} \vdash_h t(\Gamma) \Rightarrow t(\Delta)$.*

A bridge to import properties from G_{EL} to G_{PAL}.

Labelled Sequent Calculus G_{PAL} for **PAL**

Lemma

For any $\mathcal{L}_{\mathsf{PAL}}$-sequent $x:\varphi, \Gamma \Rightarrow \Delta$, the following hold:

1. *if* $G_{\mathsf{PAL}} \vdash x:t(\varphi), t(\Gamma) \Rightarrow t(\Delta)$, *then* $G_{\mathsf{PAL}} \vdash x:\varphi, t(\Gamma) \Rightarrow t(\Delta)$;
2. *if* $G_{\mathsf{PAL}} \vdash t(\Gamma) \Rightarrow t(\Delta), x:t(\varphi)$, *then* $G_{\mathsf{PAL}} \vdash t(\Gamma) \Rightarrow t(\Delta), x:\varphi$.

Properties of G_{PAL}

- Structural rules are admissible in G_{PAL}

$$(wl)\ \frac{\Gamma \Rightarrow \Delta}{x:\varphi, \Gamma \Rightarrow \Delta} \qquad (wr)\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, x:\varphi}$$

$$(cl)\ \frac{x:\varphi, x:\varphi, \Gamma \Rightarrow \Delta}{x:\varphi, \Gamma \Rightarrow \Delta} \qquad (cr)\ \frac{\Gamma \Rightarrow \Delta, x:\varphi, x:\varphi}{\Gamma \Rightarrow \Delta, x:\varphi}$$

$$(c_R l)\ \frac{x \sim_a y, x \sim_a y, \Gamma \Rightarrow \Delta}{x \sim_a y, \Gamma \Rightarrow \Delta} \qquad (c_R r)\ \frac{\Gamma \Rightarrow \Delta, x \sim_a y, x \sim_a y}{\Gamma \Rightarrow \Delta, x \sim_a y}$$

$$(Cut)\ \frac{\Gamma \Rightarrow \Delta, x:\varphi \quad x:\varphi, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

Properties of G_{PAL}

Theorem (Soundness and Completenss)

For any $\mathcal{L}_{PAL}$-formulas φ, $\varphi \in \mathsf{PAL}$ iff $G_{PAL} \vdash \Rightarrow \varphi$.

Theorem (Decidability)

G_{PAL} allows terminating proof search.

Note: 4 possible sources of non-terminating proof search can be eliminated with the help of semi-subformula and minimal derivation.

Properties of G_{PAL}

Definition (Semi-subformula)

Let φ be a $\mathcal{L}_{\mathsf{PAL}}$-formula. Formula ψ is called a *semi-subformulas* of φ if one of the following conditions hold:

1. ψ is a subformula of φ;
2. $\varphi = [\phi] * \chi$ and $\psi = *[\phi]\chi$, where $* \in \{\neg, K_a\}$;
3. $\varphi = [\phi](\chi_1 \star \chi_2)$ and $\psi = [\phi]\chi_1$ or $\psi = [\phi]\chi_2$, where $\star \in \{\wedge, \rightarrow\}$;
4. $\varphi = [\phi][\chi]\xi$ and $\psi = [\phi \wedge [\phi]\chi]\xi$.

Definition (Minimal derivation)

Minimal derivations are derivations where shortenings are not possible.

Advantages of the labelled sequent calculus

1. Restricted model can be generalized to models restricted to a list of formulas
2. The calculus is based on the original semantics for **PAL** and uses reduction rules transformed from reduction axioms to deal with the announcement operators.
3. The method can be directly applied to the action model logic.

Thanks!